

# SS-Injective Modules and Rings

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July 28, 2016

## Abstract

We introduce and investigate ss-injectivity as a generalization of both soc-injectivity and small injectivity. A module  $M$  is said to be ss- $N$ -injective (where  $N$  is a module) if every  $R$ -homomorphism from a semisimple small submodule of  $N$  into  $M$  extends to  $N$ . A module  $M$  is said to be ss-injective (resp. strongly ss-injective), if  $M$  is ss- $R$ -injective (resp. ss- $N$ -injective for every right  $R$ -module  $N$ ). Some characterizations and properties of (strongly) ss-injective modules and rings are given. Some results of Amin, Yuosif and Zeyada on soc-injectivity are extended to ss-injectivity. Also, we provide some new characterizations of universally mininjective rings, quasi-Frobenius rings, Artinian rings and semisimple rings.

**Key words and phrases:** Small injective rings (modules); soc-injective rings (modules); SS-Injective rings (modules); Perfect rings; quasi-Frobenius rings.

**2010 Mathematics Subject Classification:** Primary: 16D50, 16D60, 16D80 ; Secondary: 16P20, 16P40, 16L60 .

\* The results of this paper will be part of a MSc thesis of the first author, under the supervision of the second author at the University of Al-Qadisiyah.

## 1 Introduction

Throughout this paper,  $R$  is an associative ring with identity, and all modules are unitary  $R$ -modules. For a right  $R$ -module  $M$ , we write  $\text{soc}(M)$ ,  $J(M)$ ,  $Z(M)$ ,  $Z_2(M)$ ,  $E(M)$  and  $\text{End}(M)$  for the socle, the Jacobson radical, the singular submodule, the second singular submodule, the injective hull and the endomorphism ring of  $M$ , respectively. Also, we use  $S_r$ ,  $S_\ell$ ,  $Z_r$ ,  $Z_\ell$ ,  $Z_2^r$  and  $J$  to indicate the right socle, the left socle, the right singular ideal, the left singular ideal, the

right second singular ideal, and the Jacobson radical of  $R$ , respectively. For a submodule  $N$  of  $M$ , we write  $N \subseteq^{ess} M$ ,  $N \ll M$ ,  $N \subseteq^\oplus M$ , and  $N \subseteq^{max} M$  to indicate that  $N$  is an essential submodule, a small submodule, a direct summand, and a maximal submodule of  $M$ , respectively. If  $X$  is a subset of a right  $R$ -module  $M$ , the right (resp. left) annihilator of  $X$  in  $R$  is denoted by  $r_R(X)$  (resp.  $l_R(X)$ ). If  $M = R$ , we write  $r_R(X) = r(X)$  and  $l_R(X) = l(X)$ .

Let  $M$  and  $N$  be right  $R$ -modules,  $M$  is called soc- $N$ -injective if every  $R$ -homomorphism from the  $\text{soc}(N)$  into  $M$  extends to  $N$ . A right  $R$ -module  $M$  is called soc-injective, if  $M$  is soc- $R$ -injective. A right  $R$ -module  $M$  is called strongly soc-injective, if  $M$  is soc- $N$ -injective for all right  $R$ -module  $N$  [2]

Recall that a right  $R$ -module  $M$  is called mininjective [14] (resp. small injective [19], principally small injective [20]) if every  $R$ -homomorphism from any simple (resp. small, principally small) right ideal to  $M$  extend to  $R$ . A ring is called right mininjective (resp. small injective, principally small injective) ring, if it is right mininjective (resp. small injective, principally small injective) as right  $R$ -module. A ring  $R$  is called right Kasch if every simple right  $R$ -module embeds in  $R$  (see for example [15]). Recall that a ring  $R$  is called semilocal if  $R/J$  is a semisimple [11]. Also, a ring  $R$  is said to be right perfect if every right  $R$ -module has a projective cover. Recall that a ring  $R$  is said to be quasi-Frobenius (or  $QF$ ) ring if it is right (or left) artinian and right (or left) self-injective; or equivalently, every injective right  $R$ -module is projective.

In this paper, we introduce and investigate the notions of ss-injective and strongly ss-injective modules and rings. Examples are given to show that the (strong) ss-injectivity is distinct from that of mininjectivity, principally small injectivity, small injectivity, simple J-injectivity, and (strong) soc-injectivity. Some characterizations and properties of (strongly) ss-injective modules and rings are given.

W. K. Nicholson and M. F. Yousif in [14] introduced the notion of universally mininjective ring, a ring  $R$  is called right universally mininjective if  $S_r \cap J = 0$ . In Section 2, we show that  $R$  is a right universally mininjective ring if and only if every simple right  $R$ -module is ss-injective. We also prove that if  $M$  is a projective right  $R$ -module, then every quotient of an ss- $M$ -injective right  $R$ -module is ss- $M$ -injective if and only if every sum of two ss- $M$ -injective submodules of a right  $R$ -module is ss- $M$ -injective if and only if  $\text{Soc}(M) \cap J(M)$  is projective. Also, some results are given in terms of ss-injectivity modules. For example, every simple singular right  $R$ -module is ss-injective implies that  $S_r$  projective and  $r(a) \subseteq^\oplus R_R$  for all  $a \in S_r \cap J$ , and if  $M$  is a finitely generated right  $R$ -module, then  $\text{Soc}(M) \cap J(M)$  is finitely generated if and only if every direct sum of ss- $M$ -injective right  $R$ -modules is ss- $M$ -injective if and only if every direct sum of  $\mathbb{N}$  copies of ss- $M$ -injective right  $R$ -module is ss- $M$ -injective.

In Section 3, we show that a right  $R$ -module  $M$  is strongly ss-injective if and only if every small submodule  $A$  of a right  $R$ -module  $N$ , every  $R$ -homomorphism  $\alpha : A \rightarrow M$  with  $\alpha(A)$  semisimple extends to  $N$ . In particular,  $R$  is semiprimitive if every simple right  $R$ -module is strongly ss-injective, but not conversely. We also prove that if  $R$  is a right perfect ring, then a right  $R$ -module  $M$  is strongly soc-injective if and only if  $M$  is strongly ss-injective. A results ([2, Theorem 3.6 and Proposition 3.7]) are extended. We prove that a ring  $R$  is right artinian if and only if every direct sum of strongly ss-injective right  $R$ -modules is injective, and  $R$  is  $QF$  ring if and only if every strongly ss-injective right  $R$ -module is projective.

In Section 4, we extend the results ([2, Proposition 4.6 and Theorem 4.12]) from a soc-injective ring to an ss-injective ring (see Proposition 4.14 and Corollary 4.15).

In Section 5, we show that a ring  $R$  is  $QF$  if and only if  $R$  is strongly ss-injective and right noetherian with essential right socle if and only if  $R$  is strongly ss-injective,  $l(J^2)$  is countable generated left ideal,  $S_r \subseteq^{ess} R_R$ , and the chain  $r(x_1) \subseteq r(x_2 x_1) \subseteq \dots \subseteq r(x_n x_{n-1} \dots x_1) \subseteq \dots$  terminates for every infinite sequence  $x_1, x_2, \dots$  in  $R$  (see Theorem 5.10 and Theorem 5.12). Finally, we prove that a ring  $R$  is  $QF$  if and only if  $R$  is strongly left and right ss-injective, left

Kasch, and  $J$  is left  $t$ -nilpotent (see Theorem 5.15), extending a result of I. Amin, M. Yousif and N. Zeyada [2, Proposition 5.8] on strongly soc-injective rings.

General background materials can be found in [3], [9] and [10].

## 2 SS-Injective Modules

**Definition 2.1.** Let  $N$  be a right  $R$ -module. A right  $R$ -module  $M$  is said to be *ss- $N$ -injective*, if for any semisimple small submodule  $K$  of  $N$ , any right  $R$ -homomorphism  $f : K \rightarrow M$  extends to  $N$ . A module  $M$  is said to be *ss-quasi-injective* if  $M$  is *ss- $M$ -injective*.  $M$  is said to be *ss-injective* if  $M$  is *ss- $R$ -injective*. A ring  $R$  is said to be *right ss-injective* if the right  $R$ -module  $R_R$  is *ss-injective*.

**Definition 2.2.** A right  $R$ -module  $M$  is said to be *strongly ss-injective* if  $M$  is *ss- $N$ -injective*, for all right  $R$ -module  $N$ . A ring  $R$  is said to be *strongly right ss-injective* if the right  $R$ -module  $R_R$  is *strongly ss-injective*.

**Example 2.3.** (1) Every soc-injective module is ss-injective, but not conversely (see Example 5.8).

(2) Every small injective module is ss-injective, but not conversely (see Example 5.6).

(3) Every  $\mathbb{Z}$ -module is ss-injective. In fact, if  $M$  is a  $\mathbb{Z}$ -module, then  $M$  is small injective (by [19, Theorem 2.8] and hence it is ss-injective.

(4) The two classes of principally small injective rings and ss-injective rings are different (see [15, Example 5.2], Example 4.4 and Example 5.6).

(5) Every strongly soc-injective module is strongly ss-injective, but not conversely (see Example 5.8).

(6) Every strongly ss-injective module is ss-injective, but not conversely (see Example 5.7).

**Theorem 2.4.** The following statements hold:

(1) Let  $N$  be a right  $R$ -module and let  $\{M_i : i \in I\}$  be a family of right  $R$ -modules. Then the direct product  $\prod_{i \in I} M_i$  is *ss- $N$ -injective* if and only if each  $M_i$  is *ss- $N$ -injective*, for all  $i \in I$ .

(2) Let  $M$ ,  $N$  and  $K$  be right  $R$ -modules with  $K \subseteq N$ . If  $M$  is *ss- $N$ -injective*, then  $M$  is *ss- $K$ -injective*.

(3) Let  $M$ ,  $N$  and  $K$  be right  $R$ -modules with  $M \cong N$ . If  $M$  is *ss- $K$ -injective*, then  $N$  is *ss- $K$ -injective*.

(4) Let  $M$ ,  $N$  and  $K$  be right  $R$ -modules with  $K \cong N$ . If  $M$  is *ss- $K$ -injective*, then  $M$  is *ss- $N$ -injective*.

(5) Let  $M$ ,  $N$  and  $K$  be right  $R$ -modules with  $N$  is a direct summand of  $M$ . If  $M$  is *ss- $K$ -injective*, then  $N$  is *ss- $K$ -injective*.

*Proof.* Clear. □

**Corollary 2.5.** (1) If  $N$  is a right  $R$ -module, then a finite direct sum of *ss- $N$ -injective* modules is again *ss- $N$ -injective*. Moreover, a finite direct sum of *ss-injective* (resp. *strongly ss-injective*) modules is again *ss-injective* (resp. *strongly ss-injective*).

(2) A direct summand of an *ss-quasi-injective* (resp. *ss-injective*, *strongly ss-injective*) module is again *ss-quasi-injective* (resp. *ss-injective*, *strongly ss-injective*).

*Proof.* (1) By taking the index  $I$  to be a finite set and applying Theorem 2.4(1).

(2) This follows from Theorem 2.4(5). □

**Lemma 2.6.** Every *ss-injective* right  $R$ -module is *right mininjective*.

*Proof.* Let  $I$  be a simple right ideal of  $R$ . By [16, Lemma 3.8] we have that either  $I$  is nilpotent or a direct summand of  $R$ . If  $I$  is a nilpotent, then  $I \subseteq J$  by [6, Corollary 6.2.8] and hence  $I$  is a semisimple small right ideal of  $R$ . Thus every ss-injective right  $R$ -module is right mininjective.  $\square$

It easy to prove the following proposition.

**Proposition 2.7.** *Let  $N$  be a right  $R$ -module. If  $J(N)$  is a small submodule of  $N$ , then a right  $R$ -module  $M$  is ss- $N$ -injective if and only if any  $R$ -homomorphism  $f : \text{soc}(N) \cap J(N) \rightarrow M$  extends to  $N$ .*

**Proposition 2.8.** *Let  $N$  be a right  $R$ -module and  $\{A_i : i = 1, 2, \dots, n\}$  be a family of finitely generated right  $R$ -modules. Then  $N$  is ss- $\bigoplus_{i=1}^n A_i$ -injective if and only if  $N$  is ss- $A_i$ -injective, for all  $i = 1, 2, \dots, n$ .*

*Proof.*  $(\Rightarrow)$  This follows from Theorem 2.4((2),(4)).

$(\Leftarrow)$  By [5, Proposition (I.4.1) and Proposition (I.1.2)] we have  $\text{soc}(\bigoplus_{i=1}^n A_i) \cap J(\bigoplus_{i=1}^n A_i) = (\text{soc} \cap J)(\bigoplus_{i=1}^n A_i) = \bigoplus_{i=1}^n (\text{soc} \cap J)(A_i) = \bigoplus_{i=1}^n (\text{soc}(A_i) \cap J(A_i))$ . For  $j = 1, 2, \dots, n$ , consider the following diagram:

$$\begin{array}{ccc} K_j = \text{soc}(A_j) \cap J(A_j) & \xrightarrow{i_2} & A_j \\ i_{K_j} \downarrow & & \downarrow i_{A_j} \\ \bigoplus_{i=1}^n (\text{soc}(A_i) \cap J(A_i)) & \xrightarrow{i_1} & \bigoplus_{i=1}^n A_i \\ f \downarrow & & \\ N & & \end{array}$$

where  $i_1, i_2$  are inclusion maps and  $i_{K_j}, i_{A_j}$  are injection maps. By hypothesis, there exists an  $R$ -homomorphism  $h_j : A_j \rightarrow N$  such that  $h_j \circ i_2 = f \circ i_{K_j}$ , also there exists exactly one homomorphism  $h : \bigoplus_{i=1}^n A_i \rightarrow N$  satisfying  $h_j = h \circ i_{A_j}$  by [9, Theorem 4.1.6(2)]. Thus  $f \circ i_{K_j} =$

$h_j \circ i_2 = h \circ i_{A_j} \circ i_2 = h \circ i_1 \circ i_{K_j}$  for all  $j = 1, 2, \dots, n$ . Let  $(a_1, a_2, \dots, a_n) \in \bigoplus_{i=1}^n (\text{soc}(A_i) \cap J(A_i))$ , thus  $a_j \in \text{soc}(A_j) \cap J(A_j)$ , for all  $i = 1, 2, \dots, n$  and,  $f(a_1, a_2, \dots, a_n) = f(i_{K_1}(a_1)) + f(i_{K_2}(a_2)) + \dots + f(i_{K_n}(a_n)) = (h \circ i_1)(a_1, a_2, \dots, a_n)$ . Thus  $f = h \circ i_1$  and the proof is complete.  $\square$

**Corollary 2.9.** *Let  $M$  be a right  $R$ -module and  $1 = e_1 + e_2 + \dots + e_n$  in  $R$  such that  $e_i$  are orthogonal idempotent. Then  $M$  is ss-injective if and only if  $M$  is ss- $e_i R$ -injective for every  $i = 1, 2, \dots, n$ .*

(2) For idempotents  $e$  and  $f$  of  $R$ . If  $eR \cong fR$  and  $M$  is ss- $eR$ -injective, then  $M$  is ss- $fR$ -injective.

*Proof.* (1) From [3, Corollary 7.3], we have  $R = \bigoplus_{i=1}^n e_i R$ , thus it follows from Proposition 2.8 that  $M$  is ss-injective if and only if  $M$  is ss- $e_i R$ -injective for all  $1 \leq i \leq n$ .

(2) This follows from Theorem 2.4(4).  $\square$

**Proposition 2.10.** *A right  $R$ -module  $M$  is ss-injective if and only if  $M$  is ss- $P$ -injective, for every finitely generated projective right  $R$ -module  $P$ .*

*Proof.*  $(\Rightarrow)$  Let  $M$  be an ss-injective  $R$ -module, thus it follows from Proposition 2.8 that  $M$  is ss- $R^n$ -injective for any  $n \in \mathbb{Z}^+$ . Let  $P$  be a finitely generated projective  $R$ -module, thus by [1, Corollary 5.5], we have that  $P$  is a direct summand of a module isomorphic to  $R^m$  for some  $m \in \mathbb{Z}^+$ . Since  $M$  is ss- $R^m$ -injective, thus  $M$  is ss- $P$ -injective by Theorem 2.4((2),(4)).

$(\Leftarrow)$  By the fact that  $R$  is projective.  $\square$

**Proposition 2.11.** *The following statements are equivalent for a right  $R$ -module  $M$ .*

- (1) *Every right  $R$ -module is ss- $M$ -injective.*
- (2) *Every simple submodule of  $M$  is ss- $M$ -injective.*
- (3)  $\text{soc}(M) \cap J(M) = 0$ .

*Proof.* (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (1) are obvious.

(2)  $\Rightarrow$  (3) Assume that  $\text{soc}(M) \cap J(M) \neq 0$ , thus  $\text{soc}(M) \cap J(M) = \bigoplus_{i \in I} x_i R$  where  $x_i R$  is a simple small submodule of  $M$ , for each  $i \in I$ . Therefore,  $x_i R$  is ss- $M$ -injective for each  $i \in I$  by hypothesis. For any  $i \in I$ , the inclusion map from  $x_i R$  to  $M$  is split, so we have that  $x_i R$  is a direct summand of  $M$ . Since  $x_i R$  is small submodule of  $M$ , thus  $x_i R = 0$  and hence  $x_i = 0$  for all  $i \in I$  and this a contradiction.  $\square$

**Lemma 2.12.** *Let  $M$  be an ss-quasi-injective right  $R$ -module and  $S = \text{End}(M_R)$ , then the following statements hold:*

- (1)  $l_M r_R(m) = Sm$  for all  $m \in \text{soc}(M) \cap J(M)$ .
- (2)  $r_R(m) \subseteq r_R(n)$ , where  $m \in \text{soc}(M) \cap J(M)$ ,  $n \in M$  implies  $Sn \subseteq Sm$ .
- (3)  $l_S(mR \cap r_M(\alpha)) = l_S(m) + S\alpha$ , where  $m \in \text{soc}(M) \cap J(M)$ ,  $\alpha \in S$ .
- (4) *If  $kR$  is a simple submodule of  $M$ , then  $Sk$  is a simple left  $S$ -module, for all  $k \in J(M)$ . Moreover,  $\text{soc}(M) \cap J(M) \subseteq \text{soc}({}_S M)$ .*
- (5)  $\text{soc}(M) \cap J(M) \subseteq r_M(J({}_S S))$ .
- (6)  $l_S(A \cap B) = l_S(A) + l_S(B)$ , for every semisimple small right submodules  $A$  and  $B$  of  $M$ .

*Proof.* (1) Let  $n \in l_M r_R(m)$ , thus  $r_R(m) \subseteq r_R(n)$ . Now, let  $\gamma: mR \rightarrow M$  is given by  $\gamma(mr) = nr$ , thus  $\gamma$  is a well define  $R$ -homomorphism. By hypothesis, there exists an endomorphism  $\beta$  of  $M$  such that  $\beta|_{mR} = \gamma$ . Therefore,  $n = \gamma(m) = \beta(m) \in Sm$ , that is  $l_M r_R(m) \subseteq Sm$ . The inverse inclusion is clear.

(2) Let  $n \in M$  and  $m \in \text{soc}(M) \cap J(M)$ . Since  $r_R(m) \subseteq r_R(n)$ , then  $n \in l_M r_R(m)$ . By (1), we have  $n \in Sm$  as desired.

(3) If  $f \in l_S(m) + S\alpha$ , then  $f = f_1 + f_2$  such that  $f_1(m) = 0$  and  $f_2 = g\alpha$ , for some  $g \in S$ . For all  $n \in mR \cap r_M(\alpha)$ , we have  $n = mr$  and  $\alpha(n) = 0$  for some  $r \in R$ . Since  $f_1(n) = f_1(mr) = f_1(m)r = 0$  and  $f_2(n) = g(\alpha(n)) = g(0) = 0$ , thus  $f \in l_S(mR \cap r_M(\alpha))$  and this implies that  $l_S(m) + S\alpha \subseteq l_S(mR \cap r_M(\alpha))$ . Now, we will prove that the other inclusion. Let  $g \in l_S(mR \cap r_M(\alpha))$ . If  $r \in r_R(\alpha(m))$ , then  $\alpha(mr) = 0$ , so  $mr \in mR \cap r_M(\alpha)$  which yields  $r_R(\alpha(m)) \subseteq r_R(g(m))$ . Since  $m \in \text{soc}(M) \cap J(M)$ , thus  $\alpha(m) \in \text{soc}(M) \cap J(M)$ . By (2), we have that  $g(m) = \gamma\alpha(m)$  for some  $\gamma \in S$ . Therefore,  $g - \gamma\alpha \in l_S(m)$  which leads to  $g \in l_S(m) + S\alpha$ . Thus  $l_S(mR \cap r_M(\alpha)) = l_S(m) + S\alpha$ .

(4) To prove  $Sk$  is simple left  $S$ -module, we need only show that  $Sk$  is cyclic for any nonzero element in it. If  $0 \neq \alpha(k) \in Sk$ , then  $\alpha: kR \rightarrow \alpha(kR)$  is an  $R$ -isomorphism. Since  $\alpha \in S$ , then  $\alpha(kR) \ll M$ . Since  $M$  is ss-quasi-injective, thus  $\alpha^{-1}: \alpha(kR) \rightarrow kR$  has an extension  $\beta \in S$  and hence  $\beta(\alpha(k)) = \alpha^{-1}(\alpha(k)) = k$ , so  $k \in S\alpha k$  which leads to  $Sk = S\alpha k$ . Therefore  $Sk$  is a simple left  $S$ -module and this leads to  $\text{soc}(M) \cap J(M) \subseteq \text{soc}({}_S M)$ .

(5) If  $mR$  is simple and small submodule of  $M$ , then  $m \neq 0$ . We claim that  $\alpha(m) = 0$  for all  $\alpha \in J(S)$ , thus  $mR \subseteq r_M(J(S))$ . Otherwise,  $\alpha(m) \neq 0$  for some  $\alpha \in J(S)$ . Thus  $\alpha: mR \rightarrow \alpha(mR)$  is an  $R$ -isomorphism. Now, we need prove that  $r_R(\alpha(m)) = r_R(m)$ . Let  $r \in r_R(m)$ , so  $\alpha(m)r = \alpha(mr) = \alpha(0) = 0$  which leads to  $r_R(m) \subseteq r_R(\alpha(m))$ . The other inclusion, if  $r \in r_R(\alpha(m))$ , then  $\alpha(mr) = 0$ , that is  $mr \in \ker(\alpha) = 0$ , so  $r \in r_R(m)$ . Hence  $r_R(\alpha(m)) = r_R(m)$ . Since  $m, \alpha(m) \in \text{soc}(M) \cap J(M)$ , thus  $S\alpha m = Sm$  (by(2)) and this implies that  $m = \beta\alpha(m)$  for some  $\beta \in S$ , so  $(1 - \beta\alpha)(m) = 0$ . Since  $\alpha \in J(S)$ , then the element  $\beta\alpha$  is quasi-regular by [3, Theorem 15.3]. Thus  $1 - \beta\alpha$  is invertible and hence  $m = 0$  which is a contradiction. This shows that  $\text{soc}(M) \cap J(M) \subseteq r_M(J(S))$ .

(6) Let  $\alpha \in l_S(A \cap B)$  and consider  $f : A + B \rightarrow M$  is given by  $f(a + b) = \alpha(a)$ , for all  $a \in A$  and  $b \in B$ . Since  $M$  is ss-quasi-injective, thus there exists  $\beta \in S$  such that  $f(a + b) = \beta(a + b)$ . Thus  $\beta(a + b) = \alpha(a)$ , so  $(\alpha - \beta)(a) = \beta(b)$  which yields  $\alpha - \beta \in l_S(A)$ . Therefore,  $\alpha = \alpha - \beta + \beta \in l_S(A) + l_S(B)$  and this implies that  $l_S(A \cap B) \subseteq l_S(A) + l_S(B)$ . The other inclusion is trivial and the proof is complete.  $\square$

**Remark 2.13.** Let  $M$  be a right  $R$ -module, then  $D(S) = \{\alpha \in S = \text{End}(M) \mid r_M(\alpha) \cap mR \neq 0 \text{ for each } 0 \neq m \in \text{soc}(M) \cap J(M)\}$  is a left ideal in  $S$ .

*Proof.* This is obvious.  $\square$

**Proposition 2.14.** Let  $M$  be an ss-quasi-injective right  $R$ -module. Then  $r_M(\alpha) \subsetneq r_M(\alpha - \alpha\gamma\alpha)$ , for all  $\alpha \notin D(S)$  and for some  $\gamma \in S$ .

*Proof.* For all  $\alpha \notin D(S)$ . By hypothesis, we can find  $0 \neq m \in \text{soc}(M) \cap J(M)$  such that  $r_M(\alpha) \cap mR = 0$ . Clearly,  $r_R(\alpha(m)) = r_R(m)$ , so  $Sm = S\alpha m$  by Lemma 2.12(2). Thus  $m = \gamma\alpha m$  for some  $\gamma \in S$  and this implies that  $(\alpha - \alpha\gamma\alpha)m = 0$ . Therefore,  $m \in r_M(\alpha - \alpha\gamma\alpha)$ , but  $m \notin r_M(\alpha)$  and hence the inclusion is strictly.  $\square$

**Proposition 2.15.** Let  $M$  be an ss-quasi-injective right  $R$ -module, then the set  $\{\alpha \in S = \text{End}(M) \mid 1 - \beta\alpha \text{ is monomorphism for all } \beta \in S\}$  is contained in  $D(S)$ . Moreover,  $J({}_S S) \subseteq D(S)$ .

*Proof.* Let  $\alpha \notin D(S)$ , then there exists  $0 \neq m \in \text{soc}(M) \cap J(M)$  such that  $r_M(\alpha) \cap mR = 0$ . If  $r \in r_R(\alpha(m))$ , then  $\alpha(mr) = 0$  and so  $mr \in r_M(\alpha)$ . Since  $r_M(\alpha) \cap mR = 0$ . Thus  $r \in r_R(m)$  and hence  $r_R(\alpha(m)) \subseteq r_R(m)$ , so  $Sm \subseteq S\alpha m$  by Lemma 2.12(2). Therefore,  $m \in \ker(1 - \gamma\alpha)$  for some  $\gamma \in S$ . Since  $m \neq 0$ , thus  $1 - \gamma\alpha$  is not monomorphism and hence the inclusion holds. Now, let  $\alpha \in J({}_S S)$  we have  $\beta\alpha$  is a quasi-regular element by [3, Theorem 15.3] and hence  $1 - \beta\alpha$  is isomorphism for all  $\beta \in S$ , which completes the proof.  $\square$

**Theorem 2.16.** (ss-Baer's condition) The following statements are equivalent for a ring  $R$ .

- (1)  $M$  is an ss-injective right  $R$ -module.
- (2) If  $S_r \cap J = A \oplus B$  and  $\alpha : A \rightarrow M$  is an  $R$ -homomorphism, then there exists  $m \in M$  such that  $\alpha(a) = ma$  for all  $a \in A$  and  $mB = 0$ .
- (3) If  $S_r \cap J = A \oplus B$ , and  $\alpha : A \rightarrow M$  is an  $R$ -homomorphism, then there exists  $m \in M$  such that  $\alpha(a) = ma$ , for all  $a \in A$  and  $mB = 0$ .

*Proof.* (1) $\Rightarrow$ (2) Define  $\gamma : S_r \cap J \rightarrow M$  by  $\gamma(a + b) = \alpha(a)$  for all  $a \in A, b \in B$ . By hypothesis, there is a right  $R$ -homomorphism  $\beta : R \rightarrow M$  is an extension of  $\gamma$ , so if  $m = \beta(1)$ , then  $\alpha(a) = \gamma(a) = \beta(a) = \beta(1)a = ma$ , for all  $a \in A$ . Moreover,  $mb = \beta(b) = \gamma(b) = \alpha(0) = 0$  for all  $b \in B$ , so  $mB = 0$ .

(2) $\Rightarrow$ (1) Let  $\alpha : I \rightarrow M$  be any right  $R$ -homomorphism, where  $I$  is any semisimple small right ideal in  $R$ . By (2), there exists  $m \in M$  such that  $\alpha(a) = ma$  for all  $a \in I$ . Define  $\beta : R_R \rightarrow M$  by  $\beta(r) = mr$  for all  $r \in R$ , thus  $\beta$  extends  $\alpha$ .

(2) $\Leftrightarrow$ (3) Clear.  $\square$

A ring  $R$  is called right universally mininjective ring if it satisfies the condition  $S_r \cap J = 0$  (see for example [14]). In the next results, we give new characterizations of universally mininjective ring in terms of ss-injectivity and soc-injectivity.

**Corollary 2.17.** The following are equivalent for a ring  $R$ .

- (1)  $R$  is right universally mininjective.
- (2)  $R$  is right mininjective and every quotient of a soc-injective right  $R$ -module is soc-injective.
- (3)  $R$  is right mininjective and every quotient of an injective right  $R$ -module is soc-injective.

(4)  $R$  is right mininjective and every semisimple submodule of a projective right  $R$ -module is projective.

(5) Every right  $R$ -module is ss-injective.

(6) Every simple right ideal is ss-injective.

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) By [14, Lemma 5.1] and [2, Corollary 2.9].

(1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) By Proposition 2.11.  $\square$

**Theorem 2.18.** *If  $M$  is a projective right  $R$ -module. Then the following statements are equivalent.*

(1) Every quotient of an ss- $M$ -injective right  $R$ -module is ss- $M$ -injective.

(2) Every quotient of a soc- $M$ -injective right  $R$ -module is ss- $M$ -injective.

(3) Every quotient of an injective right  $R$ -module is ss- $M$ -injective.

(4) Every sum of two ss- $M$ -injective submodules of a right  $R$ -module is ss- $M$ -injective.

(5) Every sum of two soc- $M$ -injective submodules of a right  $R$ -module is ss- $M$ -injective.

(6) Every sum of two injective submodules of a right  $R$ -module is ss- $M$ -injective.

(7) Every semisimple small submodule of  $M$  is projective.

(8) Every simple small submodule of  $M$  is projective.

(9)  $\text{soc}(M) \cap J(M)$  is projective.

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6) and (9) $\Rightarrow$ (7) $\Rightarrow$ (8) are obvious.

(8) $\Rightarrow$ (9) Since  $\text{soc}(M) \cap J(M)$  is a direct sum of simple submodules of  $M$  and since every simple in  $J(M)$  is small in  $M$ , thus  $\text{soc}(M) \cap J(M)$  is projective.

(3) $\Rightarrow$ (7) Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xhookrightarrow{i} & M & & \\ & & \downarrow f & & & & \\ E & \xrightarrow{h} & N & \longrightarrow & 0 & & \end{array}$$

where  $E$  and  $N$  are right  $R$ -modules,  $K$  is a semisimple small submodule of  $M$ ,  $h$  is a right  $R$ -epimorphism and  $f$  is a right  $R$ -homomorphism. We can assume that  $E$  is injective (see, e.g. [6, Proposition 5.2.10]). Since  $N$  is ss- $M$ -injective, thus  $f$  can be extended to an  $R$ -homomorphism  $g : M \rightarrow N$ . By projectivity of  $M$ , thus  $g$  can be lifted to an  $R$ -homomorphism  $\tilde{g} : M \rightarrow E$  such that  $h \circ \tilde{g} = g$ . Define  $\tilde{f} : K \rightarrow E$  is the restriction of  $\tilde{g}$  over  $K$ . Clearly,  $h \circ \tilde{f} = f$  and this implies that  $K$  is projective.

(7) $\Rightarrow$ (1) Let  $N$  and  $L$  be right  $R$ -modules with  $h : N \rightarrow L$  is an  $R$ -epimorphism and  $N$  is ss- $M$ -injective. Let  $K$  be any semisimple small submodule of  $M$  and let  $f : K \rightarrow L$  be any left  $R$ -homomorphism. By hypothesis  $K$  is projective, thus  $f$  can be lifted to  $R$ -homomorphism  $g : K \rightarrow N$  such that  $h \circ g = f$ . Since  $N$  is ss- $M$ -injective, thus there exists an  $R$ -homomorphism  $\tilde{g} : M \rightarrow N$  such that  $\tilde{g} \circ i = g$ . Put  $\beta = h \circ \tilde{g} : M \rightarrow L$ . Thus  $\beta \circ i = h \circ \tilde{g} \circ i = h \circ g = f$ . Hence  $L$  is an ss- $M$ -injective right  $R$ -module.

(1) $\Rightarrow$ (4) Let  $N_1$  and  $N_2$  be two ss- $M$ -injective submodules of a right  $R$ -module  $N$ . Thus  $N_1 + N_2$  is a homomorphic image of the direct sum  $N_1 \oplus N_2$ . Since  $N_1 \oplus N_2$  is ss- $M$ -injective, thus  $N_1 + N_2$  is ss- $M$ -injective by hypothesis.

(6) $\Rightarrow$ (3) Let  $E$  be an injective right  $R$ -module with submodule  $N$ . Let  $Q = E \oplus E$ ,  $K = \{(n, n) \mid n \in N\}$ ,  $\bar{Q} = Q/K$ ,  $H_1 = \{y + K \in \bar{Q} \mid y \in E \oplus 0\}$ ,  $H_2 = \{y + K \in \bar{Q} \mid y \in 0 \oplus E\}$ . Then  $\bar{Q} = H_1 + H_2$ . Since  $(E \oplus 0) \cap K = 0$  and  $(0 \oplus E) \cap K = 0$ , thus  $E \cong H_i$ ,  $i = 1, 2$ . Since  $H_1 \cap H_2 = \{y + K \in \bar{Q} \mid y \in N \oplus 0\} = \{y + K \in \bar{Q} \mid y \in 0 \oplus N\}$ , thus  $H_1 \cap H_2 \cong N$  under  $y \mapsto y + K$  for all  $y \in N \oplus 0$ . By hypothesis,  $\bar{Q}$  is ss- $M$ -injective. Since  $H_1$  is injective, thus  $\bar{Q} = H_1 \oplus A$  for some submodule  $A$  of  $\bar{Q}$ , so  $A \cong (H_1 + H_2)/H_1 \cong H_2/H_1 \cap H_2 \cong E/N$ . By Theorem 2.4(5),  $E/N$  is ss- $M$ -injective.  $\square$

**Corollary 2.19.** *The following statements are equivalent.*

- (1) *Every quotient of an ss-injective right  $R$ -module is ss-injective.*
- (2) *Every quotient of a soc-injective right  $R$ -module is ss-injective.*
- (3) *Every quotient of a small injective right  $R$ -module is ss-injective.*
- (4) *Every quotient of an injective right  $R$ -module is ss-injective.*
- (5) *Every sum of two ss-injective submodules of any right  $R$ -module is ss-injective.*
- (6) *Every sum of two soc-injective submodules of any right  $R$ -module is ss-injective.*
- (7) *Every sum of two small injective submodules of any right  $R$ -module is ss-injective.*
- (8) *Every sum of two injective submodules of any right  $R$ -module is ss-injective.*
- (9) *Every semisimple small submodule of any projective right  $R$ -module is projective.*
- (10) *Every semisimple small submodule of any finitely generated projective right  $R$ -module is projective.*
- (11) *Every semisimple small submodule of  $R_R$  is projective.*
- (12) *Every simple small submodule of  $R_R$  is projective.*
- (13)  *$S_r \cap J$  is projective.*
- (14)  *$S_r$  is projective.*

*Proof.* The equivalence of (1), (2), (4), (5), (6), (8), (11), (12) and (13) is from Theorem 2.18.

(1) $\Rightarrow$ (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (7) $\Rightarrow$ (8) and (9) $\Rightarrow$ (10) $\Rightarrow$ (13) are clear.

(14) $\Rightarrow$ (9) By [2, Corollary 2.9].

(13) $\Rightarrow$ (14) Let  $S_r = (S_r \cap J) \oplus A$ , where  $A = \bigoplus_{i \in I} S_i$  and  $S_i$  is a right simple and summand of  $R_R$  for all  $i \in I$ . Thus  $A$  is projective, but  $S_r \cap J$  is projective, so it follows that  $S_r$  is projective.  $\square$

**Theorem 2.20.** *If every simple singular right  $R$ -module is ss-injective, then  $r(a) \subseteq^\oplus R_R$  for every  $a \in S_r \cap J$  and  $S_r$  is projective.*

*Proof.* Let  $a \in S_r \cap J$  and let  $A = RaR + r(a)$ . Thus there exists a right ideal  $B$  of  $R$  such that  $A \oplus B \subseteq^{ess} R_R$ . Suppose that  $A \oplus B \neq R_R$ , thus we choose  $I \subseteq^{max} R_R$  such that  $A \oplus B \subseteq I$  and so  $I \subseteq^{ess} R_R$ . By hypothesis,  $R/I$  is a right ss-injective. Consider the map  $\alpha : aR \rightarrow R/I$  is given by  $\alpha(ar) = r + I$  which is a well-defined  $R$ -homomorphism. Thus there exists  $c \in R$  such that  $1 + I = ca + I$  and hence  $1 - ca \in I$ . But  $ca \in RaR \subseteq I$  which leads to  $1 \in I$ , a contradiction. Thus  $A \oplus B = R$  and hence  $RaR + (r(a) \oplus B) = R$ . Since  $RaR \ll R_R$ , thus  $r(a) \subseteq^\oplus R_R$ . Put  $r(a) = (1 - e)R$ , for some  $e^2 = e \in R$ , so it follows that  $ax = aex$  for all  $x \in R$  and hence  $aR = aeR$ . Let  $\gamma : eR \rightarrow aeR$  be defined by  $\gamma(er) = aer$  for all  $r \in R$ . Then  $\gamma$  is a well-defined  $R$ -epimorphism. Clearly,  $\ker(\gamma) = eR \cap r(a)$ . Hence  $\gamma$  is an isomorphism and so  $aR$  is projective. Since  $S_r \cap J$  is a direct sum of simple small right ideals, thus  $S_r \cap J$  is projective and it follows from Corollary 2.19 that  $S_r$  is projective.  $\square$

**Corollary 2.21.** *The following statements are equivalent for a ring  $R$ .*

- (1)  *$R$  is right mininjective and every simple singular right  $R$ -module is ss-injective.*
- (2)  *$R$  is right universally mininjective.*

*Proof.* By Theorem 2.20 and [14, Lemma 5.1].  $\square$

Recall that a ring  $R$  is called zero insertive, if  $aRb = 0$  for each  $a, b \in R$  with  $ab = 0$  (see [19]). Note that if  $R$  is zero insertive ring, then  $RaR + r(a) \subseteq^{ess} R_R$  for every  $a \in R$  (see [19, Lemma 2.11]).

**Proposition 2.22.** *Let  $R$  be a zero insertive ring. If every simple singular right  $R$ -module is ss-injective, then  $R$  is right universally mininjective.*



*Proof.* Let  $a \in S_r \cap J$ . We claim that  $RaR + r(a) = R$ , thus  $r(a) = R$  (since  $RaR \ll R$ ), so  $a = 0$  and this means that  $S_r \cap J = 0$ . Otherwise, if  $RaR + r(a) \subsetneq R$ , then there exists a maximal right ideal  $I$  of  $R$  such that  $RaR + r(a) \subseteq I$ . Since  $I \subseteq^{ess} R_R$ , thus  $R/I$  is ss-injective by hypothesis. Consider  $\alpha : aR \rightarrow R/I$  is given by  $\alpha(ar) = r + I$  for all  $r \in R$  which is a well-defined  $R$ -homomorphism. Thus  $1 + I = ca + I$  for some  $c \in R$ . Since  $ca \in RaR \subseteq I$ , thus  $1 \in I$  and this contradicts with a maximality of  $I$ , so we must have  $RaR + r(a) = R$  and this completes the proof.  $\square$

**Theorem 2.23.** *If  $M$  is a finitely generated right  $R$ -module, then the following statements are equivalent.*

- (1)  $\text{soc}(M) \cap J(M)$  is a Noetherian  $R$ -module.
- (2)  $\text{soc}(M) \cap J(M)$  is finitely generated.
- (3) Any direct sum of ss- $M$ -injective right  $R$ -modules is ss- $M$ -injective.
- (4) Any direct sum of soc- $M$ -injective right  $R$ -modules is ss- $M$ -injective.
- (5) Any direct sum of injective right  $R$ -modules is ss- $M$ -injective.
- (6)  $K^{(S)}$  is ss- $M$ -injective for every injective right  $R$ -module  $K$  and for any index set  $S$ .
- (7)  $K^{(\mathbb{N})}$  is ss- $M$ -injective for every injective right  $R$ -module  $K$ .

*Proof.* (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) Clear.

(2) $\Rightarrow$ (3) Let  $E = \bigoplus_{i \in I} M_i$  be a direct sum of ss- $M$ -injective right  $R$ -modules and  $f : N \rightarrow E$  be a right  $R$ -homomorphism, where  $N$  is a semisimple small submodule of  $M$ . Since  $\text{soc}(M) \cap J(M)$  is finitely generated, thus  $N$  is finitely generated and hence  $f(N) \subseteq \bigoplus_{j \in I_1} M_j$ , for some finite subset  $I_1$  of  $I$ . Since a finite direct sums of ss- $M$ -injective right  $R$ -modules is ss- $M$ -injective, thus  $\bigoplus_{j \in I_1} M_j$  is ss- $M$ -injective and hence  $f$  can be extended to an  $R$ -homomorphism  $g : M \rightarrow E$ . Thus  $E$  is ss- $M$ -injective.

(7) $\Rightarrow$ (1) Let  $N_1 \subseteq N_2 \subseteq \dots$  be a chain of submodules of  $\text{soc}(M) \cap J(M)$ . For each  $i \geq 1$ , let  $E_i = E(M/N_i)$ ,  $E = \bigoplus_{i=1}^{\infty} E_i$  and  $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus (\prod_{j=1, j \neq i}^{\infty} E_j)$ , then  $M_i$  is injective. By hypothesis,

$\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} E_i) \oplus (\bigoplus_{i=1}^{\infty} \prod_{j=1, j \neq i}^{\infty} E_j)$  is ss- $M$ -injective, so it follows from Theorem 2.4(5) that  $E$  it self is

ss- $M$ -injective. Define  $f : U = \bigcup_{i=1}^{\infty} N_i \rightarrow E$  by  $f(m) = (m + N_i)_i$ . It is clear that  $f$  is a well defined  $R$ -homomorphism. Since  $M$  is finitely generated, thus  $\text{soc}(M) \cap J(M)$  is a semisimple small submodule of  $M$  and hence  $\bigcup_{i=1}^{\infty} N_i$  is a semisimple small submodule of  $M$ , so  $f$  can be extended to a right  $R$ -homomorphism  $g : M \rightarrow E$ . Since  $M$  is finitely generated, we have  $g(M) \subseteq \bigoplus_{i=1}^n E(M/N_i)$  for some  $n$  and hence  $f(\bigcup_{i=1}^{\infty} N_i) \subseteq \bigoplus_{i=1}^n E(M/N_i)$ . Since  $\pi_i f(x) = \pi_i(x + N_j)_{j \geq 1} = x + N_i$ , for all  $x \in U$  and  $i \geq 1$ , where  $\pi_i : \bigoplus_{j \geq 1} E(M/N_j) \rightarrow E(M/N_i)$  be the projection map, thus  $\pi_i f(U) = U/N_i$  for all  $i \geq 1$ . Since  $f(U) \subseteq \bigoplus_{i=1}^n E(M/N_i)$ , thus  $U/N_i = \pi_i f(U) = 0$ , for all  $i \geq n+1$ , so  $U = N_i$  for all  $i \geq n+1$  and hence the chain  $N_1 \subseteq N_2 \subseteq \dots$  terminates at  $N_{n+1}$ . Thus  $\text{soc}(M) \cap J(M)$  is a Noetherian  $R$ -module.  $\square$

**Corollary 2.24.** *If  $N$  is a finitely generated right  $R$ -module, then the following statements are equivalent.*

- (1)  $\text{soc}(N) \cap J(N)$  is finitely generated.
- (2)  $M^{(S)}$  is ss- $N$ -injective for every soc- $N$ -injective right  $R$ -module  $M$  and for any index set  $S$ .
- (3)  $M^{(S)}$  is ss- $N$ -injective for every ss- $N$ -injective right  $R$ -module  $M$  and for any index set  $S$ .

- (4)  $M^{(\mathbb{N})}$  is ss- $N$ -injective for every soc- $N$ -injective right  $R$ -module  $M$ .  
(5)  $M^{(\mathbb{N})}$  is ss- $N$ -injective for every ss- $N$ -injective right  $R$ -module  $M$ .

*Proof.* By Theorem 2.23. □

**Corollary 2.25.** *The following statements are equivalent.*

- (1)  $S_r \cap J$  is finitely generated.
- (2) Any direct sum of ss-injective right  $R$ -modules is ss-injective.
- (3) Any direct sum of soc-injective right  $R$ -modules is ss-injective.
- (4) Any direct sum of small injective right  $R$ -modules is ss-injective.
- (5) Any direct sum of injective right  $R$ -modules is ss-injective.
- (6)  $M^{(S)}$  is ss-injective for every injective right  $R$ -module  $M$  and for any index set  $S$ .
- (7)  $M^{(S)}$  is ss-injective for every soc-injective right  $R$ -module  $M$  and for any index set  $S$ .
- (8)  $M^{(S)}$  is ss-injective for every small injective right  $R$ -module  $M$  and for any index set  $S$ .
- (9)  $M^{(S)}$  is ss-injective for every ss-injective right  $R$ -module  $M$  and for any index set  $S$ .
- (10)  $M^{(\mathbb{N})}$  is ss-injective for every injective right  $R$ -module  $M$ .
- (11)  $M^{(\mathbb{N})}$  is ss-injective for every soc-injective right  $R$ -module  $M$ .
- (12)  $M^{(\mathbb{N})}$  is ss-injective for every small injective right  $R$ -module  $M$ .
- (13)  $M^{(\mathbb{N})}$  is ss-injective for every ss-injective right  $R$ -module  $M$ .

*Proof.* By applying Theorem 2.23 and Corollary 2.24. □

**Remark 2.26.** Let  $M$  be a right  $R$ -module. We denote that  $r_u(N) = \{a \in S_r \cap J \mid Na = 0\}$  and  $l_M(K) = \{m \in M \mid mK = 0\}$  where  $N \subseteq M$  and  $K \subseteq S_r \cap J$ . Clearly,  $r_u(N) \subseteq (S_r \cap J)_R$  and  $l_M(K) \subseteq {}_S M$ , where  $S = \text{End}(M_R)$  and we have the following:

- (1)  $N \subseteq l_M r_u(N)$  for all  $N \subseteq M$ .
- (2)  $K \subseteq r_u l_M(K)$  for all  $K \subseteq S_r \cap J$ .
- (3)  $r_u l_M r_u(N) = r_u(N)$  for all  $N \subseteq M$ .
- (4)  $l_M r_u l_M(K) = l_M(K)$  for all  $K \subseteq S_r \cap J$ .

*Proof.* This is clear □

**Lemma 2.27.** *The following statements are equivalent for a right  $R$ -module  $M$ :*

- (1)  $R$  satisfies the ACC for right ideals of form  $r_u(N)$ , where  $N \subseteq M$ .
- (2)  $R$  satisfies the DCC for  $l_M(K)$ , where  $K \subseteq S_r \cap J$ .
- (3) For each semisimple small right ideal  $I$  there exists a finitely generated right ideal  $K \subseteq I$  such that  $l_M(I) = l_M(K)$ .

*Proof.* (1) $\Leftrightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Consider  $\Omega = \{l_M(A) \mid A \text{ is finitely generated right ideal and } A \subseteq I\}$  which is non empty set because  $M \in \Omega$ . Now, let  $K$  be a finitely generated right ideal of  $R$  and contained in  $I$ . such that  $l_M(K)$  is minimal in  $\Omega$ . Put  $B = K + xR$ , where  $x \in I$ . Thus  $B$  is a finitely generated right ideal contained in  $I$  and  $l_M(B) \subseteq l_M(K)$ . But since  $l_M(K)$  is minimal in  $\Omega$ , thus  $l_M(B) = l_M(K)$  which yields  $l_M(K)x = 0$  for all  $x \in I$ . Therefore,  $l_M(K)I = 0$  and hence  $l_M(K) \subseteq l_M(I)$ . But  $l_M(I) \subseteq l_M(K)$ , so  $l_M(I) = l_M(K)$ .

(3) $\Rightarrow$ (1) Suppose that  $r_u(M_1) \subseteq r_u(M_2) \subseteq \dots \subseteq r_u(M_n) \subseteq \dots$ , where  $M_i \subseteq M$  for each  $i$ . Put  $D_i = l_M r_u(M_i)$  for each  $i$ , and  $I = \bigcup_{i=1}^{\infty} r_u(M_i)$ , then  $I \subseteq S_r \cap J$ . By hypothesis, there exists a finitely generated right ideal  $K$  of  $R$  and contained in  $I$  such that  $l_M(I) = l_M(K)$ . Since  $K$  is a finitely generated, thus there exists  $t \in \mathbb{N}$  such that  $K \subseteq r_u(M_n)$  for all  $n \geq t$ , that is  $l_M(K) \supseteq l_M r_u(M_n) = D_n$  for all  $n \geq t$ . Since  $l_M(K) = l_M(I) = l_M(\bigcup_{i=1}^{\infty} r_u(M_i)) = \bigcap_{i=1}^{\infty} l_M r_u(M_i) = \bigcap_{i=1}^{\infty} D_i \subseteq D_n$ , thus  $l_M(K) = D_n$  for all  $n \geq t$ . Since  $D_n = l_M r_u(M_n)$ , thus  $r_u(M_n) = r_u l_M r_u(M_n) = r_u(D_n) = r_u l_M(K)$  for all  $n \geq t$ . Thus  $r_u(M_n) = r_u(M_t)$  for all  $n \geq t$ . and hence (3) implies (1), which completes the proof. □

The first part in following proposition is obtained directly by Corollary 2.25, but we will prove it by different way.

**Proposition 2.28.** *Let  $E$  be an ss-injective right  $R$ -module. Then  $E^{(\mathbb{N})}$  is ss-injective if and only if  $R$  satisfies the ACC for right ideals of form  $r_u(N)$ , where  $N \subseteq E$ .*

*Proof.* ( $\Rightarrow$ ) Suppose that  $r_u(N_1) \subsetneq r_u(N_2) \subsetneq \dots \subsetneq r_u(N_m) \subsetneq \dots$  be a strictly chain, where  $N_i \subseteq E$ . Thus we get,  $l_E r_u(N_1) \subsetneq l_E r_u(N_2) \subsetneq \dots \subsetneq l_E r_u(N_m) \subsetneq \dots$ . For each  $i \geq 1$ , so we can find  $t_i \in l_E r_u(N_i) \setminus l_E r_u(N_{i+1})$  and  $a_{i+1} \in r_u(N_{i+1})$  such that  $t_i a_{i+1} \neq 0$ . Let  $L = \bigcup_{i=1}^{\infty} r_u(N_i)$ , then for all  $\ell \in L$  there exists  $m_\ell \geq 1$  such that  $\ell \in r_u(N_i)$  for all  $i \geq m_\ell$  and this implies that  $t_i \ell = 0$  for all  $i \geq m_\ell$ . Put  $\bar{t} = (t_i)_i$ , we have  $\bar{t} \ell \in E^{(\mathbb{N})}$  for every  $\ell \in L$ . Consider  $\alpha_{\bar{t}} : L \rightarrow E^{(\mathbb{N})}$  is given by  $\alpha_{\bar{t}}(\ell) = \bar{t} \ell$ , then  $\alpha_{\bar{t}}$  is a well-define  $R$ -homomorphism. Since  $L$  is a semisimple small right ideal, thus  $\alpha_{\bar{t}}$  extends to  $\gamma : R \rightarrow E^{(\mathbb{N})}$  (by hypothesis) and hence  $\alpha_{\bar{t}}(\ell) = \bar{t} \ell = \gamma(\ell) = \gamma(1)\ell$ . Thus there exists  $k \geq 1$  such that  $t_i \ell = 0$  for all  $i \geq k$  and all  $\ell \in L$  (since  $\gamma(1) \in E^{(\mathbb{N})}$ ), but this contradicts with  $t_k a_{k+1} \neq 0$ .

( $\Leftarrow$ ) Let  $\alpha : I \rightarrow E^{(\mathbb{N})}$  be an  $R$ -homomorphism, where  $I$  is a semisimple small right ideal, thus it follows from Lemma 2.27 that there is a finitely generated right ideal  $K \subseteq I$  such that  $l_M(I) = l_M(K)$ . Since  $E^{\mathbb{N}}$  is ss-injective, thus  $\alpha = a \cdot$  for some  $a \in E^{\mathbb{N}}$ . Write  $K = \bigoplus_{i=1}^m r_i R$ , so we have  $\alpha(r_i) = a r_i \in E^{(\mathbb{N})}$ ,  $i = 1, 2, \dots, m$ . Thus there exists  $\tilde{a} \in E^{(\mathbb{N})}$  such that  $a_n r_i = \tilde{a}_n r_i$  for all  $n \in \mathbb{N}$ ,  $i = 1, 2, \dots, m$ , where  $a_n$  is the  $n$ th-coordinate of  $a$ . Since  $K$  is generated by  $\{r_1, r_2, \dots, r_m\}$ , thus  $a r = \tilde{a} r$  for all  $r \in K$ . Therefore,  $a_n - \tilde{a}_n \in l_M(K) = l_M(I)$  for all  $n \in \mathbb{N}$  which leads to  $a_n r = \tilde{a}_n r$  for all  $r \in I$  and  $n \in \mathbb{N}$ , so  $a r = \tilde{a} r$  for all  $r \in I$ . Thus there exists  $\tilde{a} \in E^{(\mathbb{N})}$  such that  $\alpha(r) = \tilde{a} r$  for all  $r \in I$  and this means that  $E^{(\mathbb{N})}$  is ss-injective.  $\square$

**Theorem 2.29.** *The following statements are equivalent for a ring  $R$ :*

- (1)  $S_r \cap J$  is finitely generated.
- (2)  $\bigoplus_{i=1}^{\infty} E(M_i)$  is ss-injective right  $R$ -module for every simple right  $R$ -modules  $M_i$ ,  $i \geq 1$ .

*Proof.* (1) $\Rightarrow$ (2) By Corollary 2.25.

(2) $\Rightarrow$ (1) Let  $I_1 \subsetneq I_2 \subsetneq \dots$  be a properly ascending chain of semisimple small right ideals of  $R$ . Clearly,  $I = \bigcup_{i=1}^{\infty} I_i \subseteq S_r \cap J$ . For every  $i \geq 1$ , there exists  $a_i \in I$ ,  $a_i \notin I_i$  and consider  $N_i/I_i \subseteq^{max} (a_i R + I_i)/I_i$ , so  $K_i = (a_i R + I_i)/N_i$  is a simple right  $R$ -module. Define  $\alpha_i : (a_i R + I_i)/I_i \rightarrow (a_i R + I_i)/N_i$  by  $\alpha_i(x + I_i) = x + N_i$  which is right  $R$ -epimorphism. Let  $E(K_i)$  be the injective hull of  $K_i$  and  $i_i : K_i \rightarrow E(K_i)$  be the inclusion map. By injectivity of  $E(K_i)$ , there exists  $\beta_i : I/I_i \rightarrow E(K_i)$  such that  $\beta_i = i_i \alpha_i$ . Since  $a_i \notin N_i$ , then  $\beta_i(a_i + I_i) = i_i(\alpha_i(a_i + I_i)) = a_i + N_i \neq 0$  for each  $i \geq 1$ . If  $b \in I$ , then there exists  $n_b \geq 1$  such that  $b \in I_i$  for all  $i \geq n_b$  and hence  $\beta_i(b + I_i) = 0$  for all  $i \geq n_b$ . Thus we can define  $\gamma : I \rightarrow \bigoplus_{i=1}^{\infty} E(K_i)$  by  $\gamma(b) = (\beta_i(b + I_i))_i$ .

Then there exists  $\tilde{\gamma} : R \rightarrow \bigoplus_{i=1}^{\infty} E(K_i)$  such that  $\tilde{\gamma}|_I = \gamma$  (by hypothesis). Put  $\tilde{\gamma}(1) = (c_i)_i$ , thus there exists  $n \geq 1$  with  $c_i = 0$  for all  $i \geq n$ . Since  $(\beta_i(b + I_i))_i = \gamma(b) = \tilde{\gamma}(b) = \tilde{\gamma}(1)b = (c_i b)_i$  for all  $b \in I$ , thus  $\beta_i(b + I_i) = c_i b$  for all  $i \geq 1$ , so it follows that  $\beta_i(b + I_i) = 0$  for all  $i \geq n$  and all  $b \in I$  and this contradicts with  $\beta_n(a_n + I_n) \neq 0$ . Hence (2) implies (1).  $\square$

### 3 Strongly SS-Injective Modules

**Proposition 3.1.** *The following statements are equivalent.*

- (1)  $M$  is a strongly ss-injective right  $R$ -module.

(2) Every  $R$ -homomorphism  $\alpha : A \longrightarrow M$  extends to  $N$ , for all right  $R$ -module  $N$ , where  $A \ll N$  and  $\alpha(A)$  is a semisimple submodule in  $M$ .

*Proof.* (2) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) Let  $A$  be a small submodule of  $N$ , and  $\alpha : A \longrightarrow M$  be an  $R$ -homomorphism with  $\alpha(A)$  is a semisimple submodule of  $M$ . If  $B = \ker(\alpha)$ , then  $\alpha$  induces an embedding  $\tilde{\alpha} : A/B \longrightarrow M$  defined by  $\tilde{\alpha}(a+B) = \alpha(a)$ , for all  $a \in A$ . Clearly,  $\tilde{\alpha}$  is well define because if  $a_1 + B = a_2 + B$  we have  $a_1 - a_2 \in B$ , so  $\alpha(a_1) = \alpha(a_2)$ , that is  $\tilde{\alpha}(a_1 + B) = \tilde{\alpha}(a_2 + B)$ . Since  $M$  is strongly ss-injective and  $A/B$  is semisimple and small in  $N/B$ , thus  $\tilde{\alpha}$  extends to an  $R$ -homomorphism  $\gamma : N/B \longrightarrow M$ . If  $\pi : N \longrightarrow N/B$  is the canonical map, then the  $R$ -homomorphism  $\beta = \gamma \circ \pi : N \longrightarrow M$  is an extension of  $\alpha$  such that if  $a \in A$ , then  $\beta(a) = \gamma \circ \pi(a) = \gamma(a+B) = \tilde{\alpha}(a+B) = \alpha(a)$  as desired.  $\square$

**Corollary 3.2.** (1) Let  $M$  be a semisimple right  $R$ -module. If  $M$  is a strongly ss-injective, then  $M$  is small injective.

(2) If every simple right  $R$ -module is strongly ss-injective, then  $R$  is semiprimitive.

*Proof.* (1) By Proposition 3.1.

(2) By (1) and applying [19, Theorem 2.8].  $\square$

**Remark 3.3.** The converse of Corollary 3.2 is not true (see Example 3.8).

**Theorem 3.4.** If  $M$  is a strongly ss-injective (or just ss- $E(M)$ -injective) right  $R$ -module, then for every semisimple small submodule  $A$  of  $M$ , there is an injective  $R$ -module  $E_A$  such that  $M = E_A \oplus T_A$  where  $T_A$  is a submodule of  $M$  with  $T_A \cap A = 0$ . Moreover, if  $A \neq 0$ , then  $E_A$  can be taken  $A \leq^{ess} E_A$ .

*Proof.* Let  $A$  be a semisimple small submodule of  $M$ . If  $A = 0$ , we are done by taking  $E_A = 0$  and  $T_A = M$ . Suppose that  $A \neq 0$  and let  $i_1, i_2$  and  $i_3$  be inclusion maps and  $D_A = E(A)$  be the injective hull of  $A$  in  $E(M)$ . Since  $M$  is strongly ss-injective, thus  $M$  is ss- $E(M)$ -injective. Since  $A$  is a semisimple small submodule of  $M$ , it follows from [9, Lemma 5.1.3(a)] that  $A$  is a semisimple small submodule in  $E(M)$  and hence there exists an  $R$ -homomorphism  $\alpha : E(M) \longrightarrow M$  such that  $\alpha i_2 i_1 = i_3$ . Put  $\beta = \alpha i_2$ , thus  $\beta : D_A \longrightarrow M$  is an extension of  $i_3$ . Since  $A \leq^{ess} D_A$ , thus  $\beta$  is a monomorphism. Put  $E_A = \beta(D_A)$ . Since  $E_A$  is an injective submodule of  $M$ , thus  $M = E_A \oplus T_A$  for some submodule  $T_A$  of  $M$ . Since  $\beta(A) = A$ , thus  $A \subseteq \beta(D_A) = E_A$  and this means that  $T_A \cap A = 0$ . Moreover, define  $\tilde{\beta} = \beta : D_A \longrightarrow E_A$ , thus  $\tilde{\beta}$  is an isomorphism. Since  $A \leq^{ess} D_A$ , thus  $\tilde{\beta}(A) \leq^{ess} E_A$ . But  $\tilde{\beta}(A) = \beta(A) = A$ , so  $A \leq^{ess} E_A$ .  $\square$

**Corollary 3.5.** If  $M$  is a right  $R$ -module has a semisimple small submodule  $A$  such that  $A \leq^{ess} M$ , then the following conditions are equivalent.

(1)  $M$  is injective.

(2)  $M$  is strongly ss-injective.

(3)  $M$  is ss- $E(M)$ -injective.

*Proof.* (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) By Theorem 3.4, we can write  $M = E_A \oplus T_A$  where  $E_A$  injective and  $T_A \cap A = 0$ . Since  $A \leq^{ess} M$ , thus  $T_A = 0$  and hence  $M = E_A$ . Therefore,  $M$  is an injective  $R$ -module.  $\square$

**Example 3.6.**  $\mathbb{Z}_4$  as  $\mathbb{Z}$ -module is not strongly ss-injective. In particular,  $\mathbb{Z}_4$  is not ss- $\mathbb{Z}_{2^\infty}$ -injective.

*Proof.* Assume that  $\mathbb{Z}_4$  is strongly ss-injective  $\mathbb{Z}$ -module. Let  $A = \langle 2 \rangle = \{0, 2\}$ . It is clear that  $A$  is a semisimple small and essential submodule of  $\mathbb{Z}_4$  as  $\mathbb{Z}$ -module. Thus by Corollary 3.5 we have that  $\mathbb{Z}_4$  is injective  $\mathbb{Z}$ -module and this is a contradiction. Thus  $\mathbb{Z}_4$  as  $\mathbb{Z}$ -module is not strongly ss-injective. Since  $E(\mathbb{Z}_{2^2}) = \mathbb{Z}_{2^\infty}$  as  $\mathbb{Z}$ -module, thus  $\mathbb{Z}_4$  is not ss- $\mathbb{Z}_{2^\infty}$ -injective, by Corollary 3.5.  $\square$

**Corollary 3.7.** *Let  $M$  be a right  $R$ -module such that  $\text{soc}(M) \cap J(M)$  is small submodule in  $M$  (in particular, if  $M$  is finitely generated). If  $M$  is strongly ss-injective, then  $M = E \oplus T$ , where  $E$  is injective and  $T \cap \text{soc}(M) \cap J(M) = 0$ . Moreover, if  $\text{soc}(M) \cap J(M) \neq 0$ , then we can take  $\text{soc}(M) \cap J(M) \leq^{ess} E$ .*

*Proof.* By taking  $A = \text{soc}(M) \cap J(M)$  and applying Theorem 3.4 □

The following example shows that the converse of Theorem 3.4 and Corollary 3.7 is not true.

**Example 3.8.** Let  $M = \mathbb{Z}_6$  as  $\mathbb{Z}$ -module. Since  $J(M) = 0$  and  $\text{soc}(M) = M$ , thus  $\text{soc}(M) \cap J(M) = 0$ . So, we can write  $M = 0 \oplus M$  with  $M \cap (\text{soc}(M) \cap J(M)) = 0$ . Let  $N = \mathbb{Z}_8$  as  $\mathbb{Z}$ -module. Since  $J(N) = \langle \bar{2} \rangle$  and  $\text{soc}(N) = \langle \bar{4} \rangle$ . Define  $\gamma: \text{soc}(N) \cap J(N) \rightarrow M$  by  $\gamma(\bar{4}) = \bar{3}$ , thus  $\gamma$  is a  $\mathbb{Z}$ -homomorphism. Assume that  $M$  is strongly ss-injective, thus  $M$  is ss- $N$ -injective, so there exists  $\mathbb{Z}$ -homomorphism  $\beta: N \rightarrow M$  such that  $\beta \circ i = \gamma$ , where  $i$  is the inclusion map from  $\text{soc}(N) \cap J(N)$  to  $N$ . Since  $\beta(J(N)) \subseteq J(M)$ , thus  $\bar{3} = \gamma(\bar{4}) = \beta(\bar{4}) \in \beta(J(N)) \subseteq J(M) = 0$  and this contradiction, so  $M$  is not strongly ss-injective  $\mathbb{Z}$ -module.

**Corollary 3.9.** *The following statements are equivalent:*

- (1)  $\text{soc}(M) \cap J(M) = 0$ , for all right  $R$ -module  $M$ .
- (2) Every right  $R$ -module is strongly ss-injective.
- (3) Every simple right  $R$ -module is strongly ss-injective.

*Proof.* By Proposition 2.11. □

Recall that a ring  $R$  is called a right  $V$ -ring ( $GV$ -ring,  $SI$ -ring, respectively) if every simple (simple singular, singular, respectively) right  $R$ -module is injective. A right  $R$ -module  $M$  is called strongly s-injective if every  $R$ -homomorphism from  $K$  to  $M$  extends to  $N$  for every right  $R$ -module  $N$ , where  $K \subseteq Z(N)$  (see [22]). A submodule  $K$  of a right  $R$ -module  $M$  is called  $t$ -essential in  $M$  (written  $K \subseteq^{tes} M$ ) if for every submodule  $L$  of  $M$ ,  $K \cap L \subseteq Z_2(M)$  implies that  $L \subseteq Z_2(M)$ ,  $M$  is said to be  $t$ -semisimple if for every submodule  $A$  of  $M$  there exists a direct summand  $B$  of  $M$  such that  $B \subseteq^{tes} A$  (see [4]). In the next results, we will give some relations between ss-injectivity and other injectivities and we provide many new equivalences of  $V$ -rings,  $GV$ -rings,  $SI$  rings and  $QF$  rings.

**Lemma 3.10.** *Let  $M/N$  be a semisimple right  $R$ -module and  $C$  any right  $R$ -module. Then every homomorphism from a right submodule (resp. a right semisimple submodule)  $A$  of  $M$  to  $C$  can be extended to a homomorphism from  $M$  to  $C$  if and only if every homomorphism from a right submodule (resp. a right semisimple submodule)  $B$  of  $N$  to  $C$  can be extended to a homomorphism from  $M$  to  $C$ .*

*Proof.*  $(\Rightarrow)$  is obtained directly.

$(\Leftarrow)$  Let  $f$  be a right  $R$ -homomorphism from a right submodule  $A$  of  $M$  to  $C$ . Since  $M/N$  is semisimple, thus there exists a right submodule  $L$  of  $M$  such that  $A + L = M$  and  $A \cap L \leq N$  (see [11, Proposition 2.1]). Thus there exists a right  $R$ -homomorphism  $g: M \rightarrow C$  such that  $g(x) = f(x)$  for all  $x \in A \cap L$ . Define  $h: M \rightarrow C$  such that for any  $x = a + \ell$ ,  $a \in A$ ,  $\ell \in L$ ,  $h(x) = f(a) + g(\ell)$ . Thus  $h$  is a well define  $R$ -homomorphism, because if  $a_1 + \ell_1 = a_2 + \ell_2$ ,  $a_i \in A$ ,  $\ell_i \in L$ ,  $i = 1, 2$ , then  $a_1 - a_2 = \ell_2 - \ell_1 \in A \cap L$ , that is  $f(a_1 - a_2) = g(\ell_2 - \ell_1)$  which leads to  $h(a_1 + \ell_1) = h(a_2 + \ell_2)$ . Thus  $h$  is a well define  $R$ -homomorphism and extension of  $f$ . □

**Corollary 3.11.** *For right  $R$ -modules  $M$  and  $N$ , then the following hold:*

- (1) *If  $M$  is finitely generated and  $M/J(M)$  is semisimple right  $R$ -module, then  $N$  is right soc- $M$ -injective if and only if  $N$  is right ss- $M$ -injective.*
- (2) *If  $M/\text{soc}(M)$  is a semisimple right  $R$ -module, then  $N$  is soc- $M$ -injective if and only if  $N$  is  $M$ -injective.*
- (3) *If  $R/S_r$  is semisimple right  $R$ -module, then  $N$  is soc-injective if and only if  $N$  is injective.*
- (4) *If  $R/S_r$  is semisimple right  $R$ -module, then  $N$  is ss-injective if and only if  $N$  is small injective.*

*Proof.* (1). ( $\Rightarrow$ ) Clear.

( $\Leftarrow$ ) Since  $N$  is a right ss- $M$ -injective, thus every homomorphism from a semisimple small submodule of  $M$  to  $N$  extends to  $M$ . Since  $M$  is finitely generated, thus  $J(M) \ll M$  and hence every homomorphism from any semisimple submodule of  $J(M)$  to  $N$  extends to  $M$ . Since  $M/J(M)$  is semisimple. Thus every homomorphism from any semisimple submodule of  $M$  to  $N$  extends to  $M$  by Lemma 3.10. Therefore  $N$  is a soc- $M$ -injective right  $R$ -module.

(2). ( $\Rightarrow$ ) Since  $N$  is soc- $M$ -injective. Thus every homomorphism from any submodule of  $\text{soc}(M)$  to  $N$  extends to  $M$ . Since  $M/\text{soc}(M)$  is semisimple, thus Lemma 3.10 implies that every homomorphism from any submodule of  $M$  to  $N$  extends to  $M$ . Hence  $N$  is  $M$ -injective.

( $\Leftarrow$ ) Clear.

(3) By (2).

(4) Since  $R/S_r$  is semisimple right  $R$ -module, thus  $J(R/S_r) = 0$ . By [9, Theorem 9.1.4(b)], we have  $J \subseteq S_r$  and hence  $J = J \cap S_r$ . Thus  $N$  is ss-injective if and only if  $N$  is small injective.  $\square$

**Corollary 3.12.** *Let  $R$  be a semilocal ring, then  $S_r \cap J$  is finitely generated if and only if  $S_r$  is finitely generated.*

*Proof.* Suppose that  $S_r \cap J$  is finitely generated. By Corollary 2.25, every direct sum of soc-injective right  $R$ -modules is ss-injective. Thus it follows from Corollary 3.11(1) and [2, Corollary 2.11] that  $S_r$  is finitely generated. The converse is clear.  $\square$

**Theorem 3.13.** *If  $R$  is a right perfect ring, then a right  $R$ -module  $M$  is strongly soc-injective if and only if  $M$  is strongly ss-injective.*

*Proof.* ( $\Rightarrow$ ) Clear.

( $\Leftarrow$ ) Let  $R$  be a right perfect ring and  $M$  be a strongly ss-injective right  $R$ -module. By [11, Theorem 3.8],  $R$  is a semilocal ring and hence by [11, Theorem 3.5], we have every right  $R$ -module  $N$  is semilocal and hence  $N/J(N)$  is semisimple right  $R$ -module. Since  $R$  is a right perfect ring, thus the Jacobson radical of every right  $R$ -module is small by [7, Theorem 4.3 and 4.4, p. 69]. Thus  $N/J(N)$  is semisimple and  $J(N) \ll N$ , for any  $N \in \text{Mod-}R$ . Since  $M$  is strongly ss-injective, thus every homomorphism from a semisimple small submodule of  $N$  to  $M$  extends to  $N$ , for every  $N \in \text{Mod-}R$ , and this implies that every homomorphism from any semisimple submodule of  $J(N)$  to  $M$  extends to  $N$ , for every  $N \in \text{Mod-}R$ . Since  $N/J(N)$  is semisimple right  $R$ -module, for every  $N \in \text{Mod-}R$ . Thus Lemma 3.10 implies that every homomorphism from any semisimple submodule of  $N$  to  $M$  extends to  $N$ , for every  $N \in \text{Mod-}R$  and hence  $M$  is strongly soc-injective.  $\square$

**Corollary 3.14.** *A ring  $R$  is  $QF$  ring if and only if every strongly ss-injective right  $R$ -module is projective.*

*Proof.* ( $\Rightarrow$ ) If  $R$  is  $QF$  ring, then  $R$  is a right perfect ring, so by Theorem 3.13 and [2, Proposition 3.7] we have every strongly ss-injective right  $R$ -module is projective.

( $\Leftarrow$ ) By hypothesis we have every injective right  $R$ -module is projective and hence  $R$  is  $QF$  ring (see for instance [6, Proposition 12.5.13]).  $\square$

**Theorem 3.15.** *The following statements are equivalent for a ring  $R$ .*

- (1) *Every direct sum of strongly ss-injective right  $R$ -modules is injective.*
- (2) *Every direct sum of strongly soc-injective right  $R$ -modules is injective.*
- (3)  *$R$  is right artinian.*

*Proof.* (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Since every direct sum of strongly soc-injective right  $R$ -modules is injective, thus  $R$  is right noetherian and right semiartinian by [2, Theorem 3.3 and Theorem 3.6], so it follows from [18, Proposition 5.2, p.189] that  $R$  is right artinian.

(3) $\Rightarrow$ (1) By hypothesis,  $R$  is right perfect and right noetherian. It follows from Theorem 3.13 and [2, Theorem 3.3] that every direct sum of strongly ss-injective right  $R$ -modules is strongly soc-injective. Since  $R$  is right semiartinian, so [2, Theorem 3.6] implies that every direct sum of strongly ss-injective right  $R$ -modules is injective.  $\square$

**Theorem 3.16.** *If  $R$  is a right  $t$ -semisimple, then a right  $R$ -module  $M$  is injective if and only if  $M$  is strongly s-injective.*

*Proof.* ( $\Rightarrow$ ) Obvious.

( $\Leftarrow$ ) Since  $M$  is strongly s-injective, thus  $Z_2(M)$  is injective by [22, Proposition 3, p.27]. Thus every homomorphism  $f : K \rightarrow M$ , where  $K \subseteq Z_2'$  extends to  $R$  by [22, Lemma 1, p.26]. Since  $R$  is a right  $t$ -semisimple, thus  $R/Z_2'$  is a right semisimple (see [4, Theorem 2.3]). So by applying Lemma 3.10, we conclude that  $M$  is injective.  $\square$

**Corollary 3.17.** *The following statements are equivalent for a ring  $R$ .*

- (1)  *$R$  is right SI and right  $t$ -semisimple.*
- (2)  *$R$  is semisimple.*

*Proof.* (1) $\Rightarrow$ (2). Since  $R$  is a right SI ring, thus every right  $R$ -module is strongly s-injective by [22, Theorem 1, p.29]. By Theorem 3.16, we have every right  $R$ -module is injective and hence  $R$  is semisimple ring.

(2) $\Rightarrow$ (1). Clear.  $\square$

**Corollary 3.18.** *If  $R$  is a right  $t$ -semisimple ring, then  $R$  is right V-ring if and only if  $R$  is right GV-ring.*

*Proof.* ( $\Rightarrow$ ). Clear.

( $\Leftarrow$ ). By [22, Proposition 5, p.28] and Theorem 3.16.  $\square$

**Corollary 3.19.** *If  $R$  is a right  $t$ -semisimple ring, then  $R/S_r$  is noetherian right  $R$ -module if and only if  $R$  is right noetherian.*

*Proof.* If  $R/S_r$  is a noetherian right  $R$ -module, thus every direct sum of injective right  $R$ -modules is strongly s-injective by [22, Proposition 6]. Since  $R$  is right  $t$ -semisimple, so it follows from Theorem 3.16 that every direct sum of injective right  $R$ -modules is injective and hence  $R$  is right noetherian. The converse is clear.  $\square$

## 4 SS-Injective Rings

We recall that the dual of a right  $R$ -module  $M$  is  $M^d = \text{Hom}_R(M, R_R)$  and clearly that  $M^d$  is a left  $R$ -module.

**Proposition 4.1.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is a right ss-injective ring.
- (2) If  $K$  is a semisimple right  $R$ -module,  $P$  and  $Q$  are finitely generated projective right  $R$ -modules,  $\beta : K \rightarrow P$  is an  $R$ -monomorphism with  $\beta(K) \ll P$  and  $f : K \rightarrow Q$  is an  $R$ -homomorphism, then  $f$  can be extended to an  $R$ -homomorphism  $h : P \rightarrow Q$ .
- (3) If  $M$  is a right semisimple  $R$ -module and  $f$  is a nonzero monomorphism from  $M$  to  $R_R$  with  $f(M) \ll R_R$ , then  $M^d = Rf$ .

*Proof.* (2) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) Since  $Q$  is finitely generated, there is an  $R$ -epimorphism  $\alpha_1 : R^n \rightarrow Q$  for some  $n \in \mathbb{Z}^+$ . Since  $Q$  is projective, there is an  $R$ -homomorphism  $\alpha_2 : Q \rightarrow R^n$  such that  $\alpha_1 \alpha_2 = I_Q$ . Define  $\tilde{\beta} : K \rightarrow \beta(K)$  by  $\tilde{\beta}(a) = \beta(a)$  for all  $a \in K$ . Since  $R$  is a right ss-injective ring (by hypothesis), it follows from Proposition 2.8 and Corollary 2.5(1) that  $R^n$  is a right ss- $P$ -injective  $R$ -module. So there exists an  $R$ -homomorphism  $h : P \rightarrow R^n$  such that  $h\tilde{\beta} = \alpha_2 f \tilde{\beta}^{-1}$ . Put  $g = \alpha_1 h : P \rightarrow Q$ . Thus  $gi = (\alpha_1 h)i = \alpha_1(\alpha_2 f \tilde{\beta}^{-1}) = f \tilde{\beta}^{-1}$  and hence  $(g\beta)(a) = g(i(\beta(a))) = (f \tilde{\beta}^{-1})(\beta(a)) = f(a)$  for all  $a \in K$ . Therefore, there is an  $R$ -homomorphism  $g : P \rightarrow Q$  such that  $g\beta = f$ .

(1) $\Rightarrow$ (3) Let  $g \in M^d$ , we have  $gf^{-1} : f(M) \rightarrow R_R$ . Since  $f(M)$  is a semisimple small right ideal of  $R$  and  $R$  is a right ss-injective ring (by hypothesis), thus  $gf^{-1} = a$  for some  $a \in R$ . Therefore,  $g = af$  and hence  $M^d = Rf$ .

(3) $\Rightarrow$ (1) Let  $f : K \rightarrow R$  be a right  $R$ -homomorphism, where  $K$  is a semisimple small right ideal of  $R$  and let  $i : K \rightarrow R$  be the inclusion map, thus by (2) we have  $K^d = Ri$  and hence  $f = ci$  in  $K^d$  for some  $c \in R$ . Thus there is  $c \in R$  such that  $f(a) = ca$  for all  $a \in K$  and this implies that  $R$  is a right ss-injective ring. □

**Example 4.2.** (1) Every universally mininjective ring is ss-injective, but not conversely (see Example 5.7).

(2) The two classes of universally mininjective rings and soc-injective rings are different (see Example 5.7 and Example 5.8).

**Corollary 4.3.** *Let  $R$  be a right ss-injective ring. Then:*

- (1)  $R$  is a right mininjective ring.
- (2)  $lr(a) = Ra$ , for all  $a \in S_r \cap J$ .
- (3)  $r(a) \subseteq r(b)$ ,  $a \in S_r \cap J$ ,  $b \in R$  implies  $Rb \subseteq Ra$ .
- (4)  $l(bR \cap r(a)) = l(b) + Ra$ , for all  $a \in S_r \cap J$ ,  $b \in R$ .
- (5)  $l(K_1 \cap K_2) = l(K_1) + l(K_2)$ , for all semisimple small right ideals  $K_1$  and  $K_2$  of  $R$ .

*Proof.* (1) By Lemma 2.6.

(2), (3), (4) and (5) are obtained by Lemma 2.12. □

The following is an example of a right mininjective ring which is not right ss-injective.

**Example 4.4.** (The Björk Example [15, Example 2.5]). Let  $F$  be a field and let  $a \mapsto \bar{a}$  be an isomorphism  $F \rightarrow \bar{F} \subseteq F$ , where the subfield  $\bar{F} \neq F$ . Let  $R$  denote the left vector space on basis  $\{1, t\}$ , and make  $R$  into an  $F$ -algebra by defining  $t^2 = 0$  and  $ta = \bar{a}t$  for all  $a \in F$ . By [15, Example 2.5] we have  $R$  is a right mininjective local ring. It is mentioned in [2, Example 4.15], that  $R$  is not right soc-injective. Since  $R$  is a local ring, thus by Corollary 3.11(1),  $R$  is not right ss-injective ring.

**Theorem 4.5.** *Let  $R$  be a right ss-injective ring. Then:*

- (1)  $S_r \cap J \subseteq Z_r$ .
- (2) If the ascending chain  $r(a_1) \subseteq r(a_2 a_1) \subseteq \dots$  terminates for any sequence  $a_1, a_2, \dots$  in  $Z_r \cap S_r$ , then  $S_r \cap J$  is right  $t$ -nilpotent and  $S_r \cap J = Z_r \cap S_r$ .



*Proof.* (1) Let  $a \in S_r \cap J$  and  $bR \cap r(a) = 0$  for any  $b \in R$ . By Corollary 4.3(4),  $l(b) + Ra = l(bR \cap r(a)) = l(0) = R$ , so  $l(b) = R$  because  $a \in J$ , implies that  $b = 0$ . Thus  $r(a) \subseteq^{ess} R_R$  and hence  $S_r \cap J \subseteq Z_r$ .

(2) For any sequence  $x_1, x_2, \dots$  in  $Z_r \cap S_r$ , we have  $r(x_1) \subseteq r(x_2x_1) \subseteq \dots$ . By hypothesis, there exists  $m \in \mathbb{N}$  such that  $r(x_m \dots x_2x_1) = r(x_{m+1}x_m \dots x_2x_1)$ . If  $x_m \dots x_2x_1 \neq 0$ , then  $(x_m \dots x_2x_1)R \cap r(x_{m+1}) \neq 0$  and hence  $0 \neq x_m \dots x_2x_1r \in r(x_{m+1})$  for some  $r \in R$ . Thus  $x_{m+1}x_m \dots x_2x_1r = 0$  and this implies that  $x_m \dots x_2x_1r = 0$ , a contradiction. Thus  $Z_r \cap S_r$  is right t-nilpotent, so  $Z_r \cap S_r \subseteq J$ . Therefore,  $S_r \cap J = Z_r \cap S_r$  by (1).  $\square$

**Proposition 4.6.** *Let  $R$  be a right ss-injective ring. Then:*

- (1) *If  $Ra$  is a simple left ideal of  $R$ , then  $\text{soc}(aR) \cap J(aR)$  is zero or simple.*
- (2)  *$rl(S_r \cap J) = S_r \cap J$  if and only if  $rl(K) = K$  for all semisimple small right ideals  $K$  of  $R$ .*

*Proof.* (1) Suppose that  $\text{soc}(aR) \cap J(aR)$  is a nonzero. Let  $x_1R$  and  $x_2R$  be any simple small right ideals of  $R$  with  $x_i \in aR$ ,  $i = 1, 2$ . If  $x_1R \cap x_2R = 0$ , then by Corollary 4.3(5)  $l(x_1) + l(x_2) = R$ . Since  $x_i \in aR$ , thus  $x_i = ar_i$  for some  $r_i \in R$ ,  $i = 1, 2$ , that is  $l(a) \subseteq l(ar_i) = l(x_i)$ ,  $i = 1, 2$ . Since  $Ra$  is a simple, then  $l(a) \subseteq^{max} R$ , that is  $l(x_1) = l(x_2) = l(a)$ . Therefore,  $l(a) = R$  and hence  $a = 0$  and this contradicts the minimality of  $Ra$ . Thus  $\text{soc}(aR) \cap J(aR)$  is simple.

(2) Suppose that  $rl(S_r \cap J) = S_r \cap J$  and let  $K$  be a semisimple small right ideal of  $R$ , trivially we have  $K \subseteq rl(K)$ . If  $K \cap xR = 0$  for some  $x \in rl(K)$ , then by Corollary 4.3(5)  $l(K \cap xR) = l(K) + l(xR) = R$ , since  $x \in rl(K) \subseteq rl(S_r \cap J) = S_r \cap J$ . If  $y \in l(K)$ , then  $yx = 0$ , that is  $y(xr) = 0$  for all  $r \in R$  and hence  $l(K) \subseteq l(xR)$ . Thus  $l(xR) = R$ , so  $x = 0$  and this means that  $K \subseteq^{ess} rl(K)$ . Since  $K \subseteq^{ess} rl(K) \subseteq rl(S_r \cap J) = S_r \cap J$ , it follows that  $K = rl(K)$ . The converse is trivial.  $\square$

**Lemma 4.7.** *The following statements are equivalent.*

- (1)  *$rl(K) = K$ , for all semisimple small right ideals  $K$  of  $R$ .*
- (2)  *$r(l(K) \cap Ra) = K + r(a)$ , for all semisimple small right ideals  $K$  of  $R$  and all  $a \in R$ .*

*Proof.* (1) $\Rightarrow$ (2). Clearly,  $K + r(a) \subseteq r(l(K) \cap Ra)$  by [3, Proposition 2.16]. Now, let  $x \in r(l(K) \cap Ra)$  and  $y \in l(aK)$ . Then  $yaK = 0$  and  $y \in l(ax)$ . Thus  $l(aK) \subseteq l(ax)$ , and so  $ax \in rl(ax) \subseteq rl(aK) = aK$ , since  $aK$  is a semisimple small right ideal of  $R$ . Hence  $ax = ak$  for some  $k \in K$ , and so  $(x - k) \in r(a)$ . This leads to  $x \in K + r(a)$ , that is  $r(l(K) \cap Ra) = K + r(a)$ .

(2) $\Rightarrow$ (1). By taking  $a = 1$ .  $\square$

Recall that a right ideal  $I$  of  $R$  is said to be lie over a summand of  $R_R$ , if there exists a direct decomposition  $R_R = A_R \oplus B_R$  with  $A \subseteq I$  and  $B \cap I \ll R_R$  (see [13]) which leads to  $I = A \oplus (B \cap I)$ .

**Lemma 4.8.** *Let  $K$  be an  $m$ -generated semisimple right ideal lies over summand of  $R_R$ . If  $R$  is right ss-injective, then every homomorphism from  $K$  to  $R_R$  can be extended to an endomorphism of  $R_R$ .*

*Proof.* Let  $\alpha : K \rightarrow R$  be a right  $R$ -homomorphism. By hypothesis,  $K = eR \oplus B$ , for some  $e^2 = e \in R$ , where  $B$  is an  $m$ -generated semisimple small right ideal of  $R$ . Now, we need prove that  $K = eR \oplus (1 - e)B$ . Clearly,  $eR + (1 - e)B$  is a direct sum. Let  $x \in K$ , then  $x = a + b$  for some  $a \in eR$ ,  $b \in B$ , so we can write  $x = a + eb + (1 - e)b$  and this implies that  $x \in eR \oplus (1 - e)B$ . Conversely, let  $x \in eR \oplus (1 - e)B$ . Thus  $x = a + (1 - e)b$ , for some  $a \in eR$ ,  $b \in B$ . We obtain  $x = a + (1 - e)b = (a - eb) + b \in eR \oplus B$ . It is obvious that  $(1 - e)B$  is an  $m$ -generated semisimple small right ideal. Since  $R$  is a right ss-injective, then there exists  $\gamma \in \text{End}(R_R)$  such that  $\gamma_{(1-e)B} = \alpha_{|(1-e)B}$ . Define  $\beta : R_R \rightarrow R_R$  by  $\beta(x) = \alpha(ex) + \gamma((1 - e)x)$ , for all  $x \in R$  which is a well defined  $R$ -homomorphism. If  $x \in K$ , then  $x = a + b$  where  $a \in eR$  and  $b \in (1 - e)B$ , so  $\beta(x) = \alpha(ex) + \gamma((1 - e)x) = \alpha(a) + \gamma(b) = \alpha(a) + \alpha(b) = \alpha(x)$  which yields  $\beta$  is an extension of  $\alpha$ .  $\square$

**Corollary 4.9.** *Let  $R$  be a semiregular ring (or just every finitely generated semisimple right ideal lies over a summand of  $R_R$ ). If  $R$  is a right ss-injective ring, then every  $R$ -homomorphism from a finitely generated semisimple right ideal to  $R$  extends to  $R$ .*

*Proof.* By [13, Theorem 2.9] and Lemma 4.8. □

**Corollary 4.10.** *Let  $S_r$  be a finitely generated and lie over a summand of  $R_R$ , then  $R$  is a right ss-injective ring if and only if  $R$  is right soc-injective.*

Recall that a ring  $R$  is called right minannihilator if every simple right ideal  $K$  of  $R$  is an annihilator; equivalently, if  $rl(K) = K$  (see [14]).

**Lemma 4.11.** *A ring  $R$  is a right minannihilator if and only if  $rl(K) = K$  for any simple small right ideal  $K$  of  $R$ .*

**Lemma 4.12.** *A ring  $R$  is a left minannihilator if and only if  $lr(K) = K$  for any simple small left ideal  $K$  of  $R$ .*

**Corollary 4.13.** *Let  $R$  be a right ss-injective ring, then the following hold:*

(1) *If  $rl(S_r \cap J) = S_r \cap J$ , then  $R$  is right minannihilator.*

(2) *If  $S_\ell \subseteq S_r$ , then:*

i)  $S_\ell = S_r$ .

ii)  $R$  is a left minannihilator ring.

*Proof.* (1) Let  $aR$  be a simple small right ideal of  $R$ , thus  $rl(a) = aR$  by Proposition 4.6(2). Therefore,  $R$  is a right minannihilator ring.

(2) i) Since  $R$  is a right ss-injective ring, thus it is right mininjective and it follows from [14, Proposition 1.14 (4)] that  $S_\ell = S_r$ .

ii) If  $Ra$  is a simple small left ideal of  $R$ , then  $lr(a) = Ra$  by Corollary 4.3(2) and hence  $R$  is a left minannihilator ring. □

**Proposition 4.14.** *The following statements are equivalent for a right ss-injective ring  $R$ .*

(1)  $S_\ell \subseteq S_r$ .

(2)  $S_\ell = S_r$ .

(3)  $R$  is a left mininjective ring.

*Proof.* (1) $\Rightarrow$ (2) By Corollary 4.13(2) (i).

(2) $\Rightarrow$ (3) By Corollary 4.13(2) and [15, Corollary 2.34], we need only show that  $R$  is right minannihilator ring. Let  $aR$  be a simple small right ideal, then  $Ra$  is a simple small left ideal by [14, Theorem 1.14]. Let  $0 \neq x \in rl(aR)$ , then  $l(a) \subseteq l(x)$ . Since  $l(a) \leq^{max} R$ , thus  $l(a) = l(x)$  and hence  $Rx$  is simple left ideal, that is  $x \in S_r$ . Now, if  $Rx = Re$  for some  $e = e^2 \in R$ , then  $e = rx$  for some  $0 \neq r \in R$ . Since  $(e - 1)e = 0$ , then  $(e - 1)rx = 0$ , that is  $(e - 1)ra = 0$  and this implies that  $ra \in eR$ . Thus  $raR \subseteq eR$ , but  $eR$  is semisimple right ideal, so  $raR \subseteq^\oplus R$  and hence  $ra = 0$ . Therefore,  $rx = 0$ , that is  $e = 0$ , a contradiction. Thus  $x \in J$  and hence  $x \in S_r \cap J$ . Therefore,  $aR \subseteq rl(aR) \subseteq S_r \cap J$ . Now, let  $aR \cap yR = 0$  for some  $y \in rl(aR)$ , thus  $l(aR) + l(yR) = l(aR \cap yR) = R$ . Since  $y \in rl(aR)$ , thus  $l(aR) \subseteq l(yR)$  and hence  $l(yR) = R$ , that is  $y = 0$ . Therefore,  $aR \subseteq^{ess} rl(aR)$ , so  $aR = rl(aR)$  as desired.

(3) $\Rightarrow$ (1) Follows from [15, Corollary 2.34]. □

Recall that a ring  $R$  is said to be right minfull if it is semiperfect, right mininjective and  $\text{soc}(eR) \neq 0$  for each local idempotent  $e \in R$  (see [15]). A ring  $R$  is called right min-PF, if it is a semiperfect, right mininjective,  $S_r \subseteq^{ess} R_R$ ,  $lr(K) = K$  for every simple left ideal  $K \subseteq Re$  for some local idempotent  $e \in R$  (see [15]).

**Corollary 4.15.** *Let  $R$  be a right ss-injective ring, semiperfect with  $S_r \subseteq^{ess} R_R$ . Then  $R$  is right minfull ring and the following statements hold:*

- (1) *Every simple right ideal of  $R$  is essential in a summand.*
- (2)  *$\text{soc}(eR)$  is simple and essential in  $eR$  for every local idempotent  $e \in R$ . Moreover,  $R$  is right finitely cogenerated.*
- (3) *For every semisimple right ideal  $I$  of  $R$ , there exists  $e = e^2 \in R$  such that  $I \subseteq^{ess} rl(I) \subseteq^{ess} eR$ .*
- (4)  *$S_r \subseteq S_\ell \subseteq rl(S_r)$ .*
- (5) *If  $I$  is a semisimple right ideal of  $R$  and  $aR$  is a simple right ideal of  $R$  with  $I \cap aR = 0$ , then  $rl(I \oplus aR) = rl(I) \oplus rl(aR)$ .*
- (6)  *$rl(\bigoplus_{i=1}^n a_i R) = \bigoplus_{i=1}^n rl(a_i R)$ , where  $\bigoplus_{i=1}^n a_i R$  is a direct sum of simple right ideals.*
- (7) *The following statements are equivalent.*
  - (a)  $S_r = rl(S_r)$ .
  - (b)  $K = rl(K)$  for every semisimple right ideals  $K$  of  $R$ .
  - (c)  $kR = rl(kR)$  for every simple right ideals  $kR$  of  $R$ .
  - (d)  $S_r = S_\ell$ .
  - (e)  $\text{soc}(Re)$  is simple for all local idempotent  $e \in R$ .
  - (f)  $\text{soc}(Re) = S_r e$  for every local idempotent  $e \in R$ .
  - (g)  $R$  is left mininjective.
  - (h)  $L = lr(L)$  for every semisimple left ideals  $L$  of  $R$ .
  - (i)  $R$  is left minfull ring.
  - (j)  $S_r \cap J = rl(S_r \cap J)$ .
  - (k)  $K = rl(K)$  for every semisimple small right ideals  $K$  of  $R$ .
  - (l)  $L = lr(L)$  for every semisimple small left ideals  $L$  of  $R$ .
- (8) *If  $R$  satisfies any condition of (7), then  $r(S_\ell \cap J) \subseteq^{ess} R_R$ .*

*Proof.* (1), (2), (3), (4), (5) and (6) are obtained by Corollary 3.11 and [2, Theorem 4.12].

(7) The equivalence of (a), (b), (c), (d), (e), (f), (g), (h) and (i) follows from Corollary 3.11 and [2, Theorem 4.12].

(b) $\Rightarrow$ (j) Clear.

(j) $\Leftrightarrow$ (k) By Proposition 4.6(2).

(k) $\Rightarrow$ (c) By Corollary 4.13(1).

(h) $\Rightarrow$ (l) Clear.

(l) $\Rightarrow$ (d) Let  $Ra$  be a simple left ideal of  $R$ . By hypothesis,  $lr(A) = A$  for any simple small left ideal  $A$  of  $R$ . By Lemma 4.12,  $lr(A) = A$  for any simple left ideal  $A$  of  $R$  and hence  $lr(Ra) = Ra$ . Thus  $R$  is a right min-PF ring and it follows from [14, Theorem 3.14] that  $S_r = S_\ell$ .

(8) Let  $K$  be a right ideal of  $R$  such that  $r(S_\ell \cap J) \cap K = 0$ . Then  $Kr(S_\ell \cap J) = 0$  and we have  $K \subseteq lr(S_\ell \cap J) = S_\ell \cap J = S_r \cap J$ . Now,  $r((S_\ell \cap J) + l(K)) = r(S_\ell \cap J) \cap K = 0$ . Since  $R$  is left Kasch, then  $(S_\ell \cap J) + l(K) = R$  by [10, Corollary 8.28(5)]. Thus  $l(K) = R$  and hence  $K = 0$ , so  $r(S_\ell \cap J) \subseteq^{ess} R_R$ .  $\square$

Recall that a right  $R$ -module  $M$  is called almost-injective if  $M = E \oplus K$ , where  $E$  is injective and  $K$  has zero radical (see [23]). After reflect on [23, Theorem 2.12] we found it is not true always and the reason is due to the homomorphism  $h : (L + J)/J \longrightarrow K$  in the part (3) $\Rightarrow$ (1) of the proof of Theorem 2.12 in [23] is not well define, in particular see the following example.

**Example 4.16.** In particular from the proof of the part (3) $\Rightarrow$ (1) in [23, Theorem 2.12], we consider  $R = \mathbb{Z}_8$  and  $M = K = \langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$ . Thus  $M = E \oplus K$ , where  $E = 0$  is a trivial injective  $R$ -module and  $J(K) = 0$ . Let  $f : L \rightarrow K$  is the identity map, where  $L = K$ . So, the map  $h : (L+J)/J \rightarrow K$  which is given by  $h(\ell+J) = f(\ell)$  is not well define, because  $J = \bar{4} + J$  but  $h(J) = f(\bar{0}) = \bar{0} \neq \bar{4} = f(\bar{4}) = h(\bar{4} + J)$ .

The following example shows that there is a contradiction in [23, Theorem 2.12].

**Example 4.17.** Assume that  $R$  is a right artinian ring but not semisimple (this claim is found because for example  $\mathbb{Z}_8$  satisfies this property). Now, let  $M$  be a simple right  $R$ -module, then  $M$  is almost-injective. Clearly,  $R$  is semilocal (see [9, Theorem 9.2.2]), thus  $M$  is injective by [23, Theorem 2.12]. Therefore,  $R$  is  $V$ -ring and hence  $R$  is a right semisimple ring but this contradiction. In other word, Since  $\mathbb{Z}_8$  is semilocal ring and  $\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$  is almost injective as  $\mathbb{Z}_8$ -module, then  $\langle \bar{4} \rangle$  is injective by [23, Theorem 2.12]. Thus  $\langle \bar{4} \rangle \subseteq^\oplus \mathbb{Z}_8$  and this contradiction.

**Theorem 4.18.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is semiprimitive and every almost-injective right  $R$ -module is quasi-continuous.
- (2)  $R$  is right ss-injective and right minannihilator ring,  $J$  is right artinian, and every almost-injective right  $R$ -module is quasi-continuous.
- (3)  $R$  is a semisimple ring.

*Proof.* (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (3) Let  $M$  be a right  $R$ -module with zero radical. If  $N$  is an arbitrary nonzero submodule of  $M$ , then  $N \oplus M$  is quasi-continuous and by [12, Corollary 2.14],  $N$  is  $M$ -injective. Thus  $N \leq^\oplus M$  and hence  $M$  is semisimple. In particular  $R/J$  is semisimple  $R$ -module and hence  $R/J$  is artinian by [9, Theorem 9.2.2(b)], so  $R$  is semilocal ring. Since  $J$  is a right artinian, then  $R$  is right artinian. So it follows from Corollary 4.15(7) that  $R$  is right and left mininjective. Thus [14, Corollary 4.8] implies that  $R$  is  $QF$  ring. By hypothesis,  $R \oplus (R/J)$  is quasi-continuous (since  $R$  is self-injective), so again by [12, Corollary 2.14] we have that  $R/J$  is injective. Since  $R$  is  $QF$  ring, then  $R/J$  is projective (see [9, Theorem 13.6.1]). Thus the canonical map  $\pi : R \rightarrow R/J$  is splits and hence  $J \leq^\oplus R$ , that is  $J = 0$ . Therefore  $R$  is semisimple.  $\square$

## 5 STRONGLY SS-INJECTIVE RINGS

**Proposition 5.1.** *A ring  $R$  is strongly right ss-injective if and only if every finitely generated projective right  $R$ -module is strongly ss-injective.*

*Proof.* Since a finite direct sum of strongly ss-injective modules is strongly ss-injective, so every finitely generated free right  $R$ -module is strongly ss-injective. But a direct summand of strongly ss-injective is strongly ss-injective. Therefore, every finitely generated projective is strongly ss-injective. The converse is clear.  $\square$

A ring  $R$  is called a right Ikeda-Nakayama ring if  $l(A \cap B) = l(A) + l(B)$  for all right ideals  $A$  and  $B$  of  $R$  (see [15, p.148]). In the next proposition, the strongly ss-injectivity gives a new version of Ikeda-Nakayama rings.

**Proposition 5.2.** *Let  $R$  be a strongly right ss-injective ring, then  $l(A \cap B) = l(A) + l(B)$  for all semisimple small right ideals  $A$  and all right ideals  $B$  of  $R$ .*

*Proof.* Let  $x \in l(A \cap B)$  and define  $\alpha : A + B \longrightarrow R_R$  by  $\alpha(a + b) = xa$  for all  $a \in A$  and  $b \in B$ . Clearly,  $\alpha$  is well define, because if  $a_1 + b_1 = a_2 + b_2$ , then  $a_1 - a_2 = b_2 - b_1$ , that is  $x(a_1 - a_2) = 0$ , so  $\alpha(a_1 + b_1) = \alpha(a_2 + b_2)$ . The map  $\alpha$  induces an  $R$ -homomorphism  $\tilde{\alpha} : (A+B)/B \longrightarrow R_R$  which is given by  $\tilde{\alpha}(a+B) = xa$  for all  $a \in A$ . Since  $(A+B)/B \subseteq \text{soc}(R/B) \cap J(R/B)$  and  $R$  is a strongly right ss-injective,  $\tilde{\alpha}$  can be extended to an  $R$ -homomorphism  $\gamma : R/B \longrightarrow R_R$ . If  $\gamma(1+B) = y$ , for some  $y \in R$ , then  $y(a+b) = xa$ , for all  $a \in A$  and  $b \in B$ . In particular,  $ya = xa$  for all  $a \in A$  and  $yb = 0$  for all  $b \in B$ . Hence  $x = (x-y) + y \in l(A) + l(B)$ . Therefore,  $l(A \cap B) \subseteq l(A) + l(B)$ . Since the converse is always holds, thus the proof is complete.  $\square$

Recall that a ring  $R$  is said to be right simple  $J$ -injective if for any small right ideal  $I$  and any  $R$ -homomorphism  $\alpha : I \longrightarrow R_R$  with simple image,  $\alpha = c \cdot$  for some  $c \in R$  (see [21]).

**Corollary 5.3.** *Every strongly right ss-injective ring is right simple  $J$ -injective.*

*Proof.* By Proposition 3.1.  $\square$

**Remark 5.4.** The converse of Corollary 5.3 is not true (see Example 5.7).

**Proposition 5.5.** *Let  $R$  be a right Kasch and strongly right ss-injective ring. Then:*

- (1)  $rl(K) = K$ , for every small right ideal  $K$ . Moreover,  $R$  is right minannihilator.
- (2) If  $R$  is left Kasch, then  $r(J) \subseteq^{ess} R_R$ .

*Proof.* (1) By Corollary 5.3 and [21, Lemma 2.4].

(2) Let  $K$  be a right ideal of  $R$  and  $r(J) \cap K = 0$ . Then  $Kr(J) = 0$  and we obtain  $K \subseteq lr(J) = J$ , because  $R$  is left Kasch. By (1), we have  $r(J + l(K)) = r(J) \cap K = 0$  and this means that  $J + l(K) = R$  (since  $R$  is left Kasch). Thus  $K = 0$  and hence  $r(J) \subseteq^{ess} R_R$ .  $\square$

The following examples show that the classes of rings: strongly ss-injective rings, soc-injective rings and of small injective rings are different.

**Example 5.6.** Let  $R = \mathbb{Z}_{(p)} = \{\frac{m}{n} \mid p \text{ does not divide } n\}$ , the localization ring of  $\mathbb{Z}$  at the prime  $p$ . Then  $R$  is a commutative local ring and it has zero socle but not principally small injective (see [20, Example 4]). Since  $S_r = 0$ , thus  $R$  is strongly soc-injective ring and hence  $R$  is strongly ss-injective ring.

**Example 5.7.** Let  $R = \left\{ \begin{pmatrix} n & x \\ 0 & n \end{pmatrix} \mid n \in \mathbb{Z}, x \in \mathbb{Z}_2 \right\}$ . Thus  $R$  is a commutative ring,  $J = S_r = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$  and  $R$  is small injective (see [19, Example(i)]). Let  $A = J$  and  $B = \left\{ \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix} \mid n \in \mathbb{Z} \right\}$ , then  $l(A) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} \mid n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\}$  and  $l(B) = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in \mathbb{Z}_2 \right\}$ . Thus  $l(A) + l(B) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} \mid n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\}$ .

Since  $A \cap B = 0$ , thus  $l(A \cap B) = R$  and this implies that  $l(A) + l(B) \neq l(A \cap B)$ . Therefore  $R$  is not strongly ss-injective and not strongly soc-injective by Proposition 5.2.

**Example 5.8.** Let  $F = \mathbb{Z}_2$  be the field of two elements,  $F_i = F$  for  $i = 1, 2, 3, \dots$ ,  $Q = \prod_{i=1}^{\infty} F_i$ ,  $S = \bigoplus_{i=1}^{\infty} F_i$ . If  $R$  is the subring of  $Q$  generated by 1 and  $S$ , then  $R$  is a Von Neumann regular ring (see [22, Example (1), p.28]). Since  $R$  is commutative, thus every simple  $R$ -module is injective by [10, Corollary 3.73]. Thus  $R$  is  $V$ -ring and hence  $J(N) = 0$  for every right  $R$ -module  $N$ . It follows from Corollary 3.9 that every  $R$ -module is strongly ss-injective. In particular,  $R$  is strongly ss-injective ring. But  $R$  is not soc-injective (see [22, Example (1)]).

**Example 5.9.** Let  $R = \mathbb{Z}_2[x_1, x_2, \dots]$  where  $\mathbb{Z}_2$  is the field of two elements,  $x_i^3 = 0$  for all  $i$ ,  $x_i x_j = 0$  for all  $i \neq j$  and  $x_i^2 = x_j^2 \neq 0$  for all  $i$  and  $j$ . If  $m = x_1^2$ , then  $R$  is a commutative, semiprimary, local, soc-injective ring with  $J = \text{span}\{m, x_1, x_2, \dots\}$ , and  $R$  has simple essential socle  $J^2 = \mathbb{Z}_2 m$  (see [2, Example 5.7]). It follows from [2, Example 5.7] that the  $R$ -homomorphism  $\gamma: J \rightarrow R$  which is given by  $\gamma(a) = a^2$  for all  $a \in J$  with simple image can be not extended to  $R$ , then  $R$  is not simple  $J$ -injective and not small injective, so it follows from Corollary 5.3 that  $R$  is not strongly ss-injective.

Recall that  $R$  is said to be right minsymmetric ring if  $aR$  is simple right ideal then  $Ra$  is simple left ideal (see [14]). Every right mininjective ring is right minsymmetric by [14, Theorem 1.14].

**Theorem 5.10.** *A ring  $R$  is QF if and only if  $R$  is a strongly right ss-injective and right noetherian ring with  $S_r \subseteq^{ess} R_R$ .*

*Proof.* ( $\Rightarrow$ ) This is clear.

( $\Leftarrow$ ) By Corollary 4.3(1),  $R$  is right minsymmetric. It follows from [19, Lemma 2.2] that  $R$  is right perfect. Thus  $R$  is strongly right soc-injective, by Theorem 3.13. Since  $S_r \subseteq^{ess} R_R$ , so it follows from [2, Corollary 3.2] that  $R$  is self-injective and hence  $R$  is QF.  $\square$

**Corollary 5.11.** *For a ring  $R$  the following statements are true.*

- (1)  *$R$  is semisimple if and only if  $S_r \subseteq^{ess} R_R$  and every semisimple right  $R$ -module is strongly soc-injective.*
- (2)  *$R$  is QF if and only if  $R$  is strongly right ss-injective, semiperfect with essential right socle and  $R/S_r$  is noetherian as right  $R$ -module.*

*Proof.* (1) Suppose that  $S_r \subseteq^{ess} R_R$  and every semisimple right  $R$ -module is strongly soc-injective, then  $R$  is a right noetherian right V-ring by [2, Proposition 3.12], so it follows from Corollary 3.9 that  $R$  is strongly right ss-injective. Thus  $R$  is QF by Theorem 5.10. But  $J = 0$ , so  $R$  is semisimple. The converse is clear.

(2) By [14, Theorem 2.9],  $J = Z_r$ . Since  $R/Z_r^r$  is a homomorphic image of  $R/Z_r$  and  $R$  is a semilocal ring, thus  $R$  is a right  $t$ -semisimple. By Corollary 3.19,  $R$  is right noetherian, so it follows from Theorem 5.10 that  $R$  is QF. The converse is clear.  $\square$

**Theorem 5.12.** *A ring  $R$  is QF if and only if  $R$  is a strongly right ss-injective,  $l(J^2)$  is a countable generated left ideal,  $S_r \subseteq^{ess} R_R$  and the chain  $r(x_1) \subseteq r(x_2 x_1) \subseteq \dots \subseteq r(x_n x_{n-1} \dots x_1) \subseteq \dots$  terminates for every infinite sequence  $x_1, x_2, \dots$  in  $R$ .*

*Proof.* ( $\Rightarrow$ ) Clear.

( $\Leftarrow$ ) By [19, Lemma 2.2],  $R$  is right perfect. Since  $S_r \subseteq^{ess} R_R$ , thus  $R$  is right Kasch (by [14, Theorem 3.7]). Since  $R$  is strongly right ss-injective, thus  $R$  is right simple  $J$ -injective, by Corollary 5.3. Now, by Proposition 5.5(1) we have  $rl(S_r \cap J) = S_r \cap J$ , so it follows from Corollary 4.15(7) that  $S_r = S_\ell$ . By [15, Lemma 3.36],  $S_r^r = l(J^2)$ . The result now follows from [21, Theorem 2.18].  $\square$

**Remark 5.13.** The condition  $S_r \subseteq^{ess} R_R$  in Theorem 5.10 and Theorem 5.12 can be not deleted, for example,  $\mathbb{Z}$  is strongly ss-injective noetherian ring but not QF.

The following two results are extension of Proposition 5.8 in [2].

**Corollary 5.14.** *The following statements are equivalent.*

- (1)  *$R$  is a QF ring.*
- (2)  *$R$  is a left perfect, strongly left and right ss-injective ring.*

*Proof.* By Corollary 5.3 and [21, Corollary 2.12]. □

**Theorem 5.15.** *The following statements are equivalent:*

- (1)  *$R$  is a QF ring.*
- (2)  *$R$  is a strongly left and right ss-injective, right Kasch and  $J$  is left  $t$ -nilpotent.*
- (3)  *$R$  is a strongly left and right ss-injective, left Kasch and  $J$  is left  $t$ -nilpotent.*

*Proof.* (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Suppose that  $xR$  is simple right ideal. Thus either  $rl(x) = xR \subseteq^{\oplus} R_R$  or  $x \in J$ . If  $x \in J$ , then  $rl(x) = xR$  (since  $R$  is right minannihilator), so it follows from Theorem 3.4 that  $rl(x) \subseteq^{ess} E \subseteq^{\oplus} R_R$ . Therefore,  $rl(x)$  is essential in a direct summand of  $R_R$  for every simple right ideal  $xR$ . Let  $K$  be a maximal left ideal of  $R$ . Since  $R$  is left Kasch, thus  $r(K) \neq 0$  by [10, Corollary 8.28]. Choose  $0 \neq y \in r(K)$ , so  $K \subseteq l(y)$  and we conclude that  $K = l(y)$ . Since  $Ry \cong R/l(y)$ , thus  $Ry$  is simple left ideal. But  $R$  is left mininjective ring, so  $yR$  is right simple ideal by [14, Theorem 1.14] and this implies that  $r(K) \subseteq^{ess} eR$  for some  $e^2 = e \in R$  (since  $r(K) = rl(y)$ ). Thus  $R$  is semiperfect by [15, Lemma 4.1] and hence  $R$  is left perfect (since  $J$  is left  $t$ -nilpotent), so it follows from Corollary 5.14 that  $R$  is QF.

(2) $\Rightarrow$ (1) It is similar to the proof of (3) $\Rightarrow$ (1). □

**Theorem 5.16.** *The ring  $R$  is QF if and only if  $R$  is strongly left and right ss-injective, left and right Kasch, and the chain  $l(a_1) \subseteq l(a_1a_2) \subseteq l(a_1a_2a_3) \subseteq \dots$  terminates for every  $a_1, a_2, \dots \in Z_\ell$ .*

*Proof.* ( $\Rightarrow$ ) Clear.

( $\Leftarrow$ ) By Proposition 5.5,  $l(J)$  is essential in  ${}_R R$ . Thus  $J \subseteq Z_\ell$ . Let  $a_1, a_2, \dots \in J$ , we have  $l(a_1) \subseteq l(a_1a_2) \subseteq l(a_1a_2a_3) \subseteq \dots$ . Thus there exists  $k \in \mathbb{N}$  such that  $l(a_1 \dots a_k) = l(a_1 \dots a_k a_{k+1})$  (by hypothesis). Suppose that  $a_1 \dots a_k \neq 0$ , so  $R(a_1 \dots a_k) \cap l(a_{k+1}) \neq 0$  (since  $l(a_{k+1})$  is essential in  ${}_R R$ ). Thus  $ra_1 \dots a_k \neq 0$  and  $ra_1 \dots a_k a_{k+1} = 0$  for some  $r \in R$ , a contradiction. Therefore,  $a_1 \dots a_k = 0$  and hence  $J$  is left  $t$ -nilpotent, so it follows from Theorem 5.15 that  $R$  is QF. □

**Corollary 5.17.** *The ring  $R$  is QF if and only if  $R$  is strongly left and right ss-injective with essential right socle, and the chain  $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq \dots$  terminates for every infinite sequence  $a_1, a_2, \dots$  in  $R$ .*

*Proof.* By [19, Lemma 2.2] and Corollary 5.14. □

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